An identity that changed the course of history

Michael D. Hirschhorn

Presented at Ramanujan 125, Gainesville, 6th November, 2012

Introduction

In his first letter to Hardy (almost 100 years ago), Ramanujan included the following three claims.

If

$$u = \frac{x}{1+}\,\frac{x^5}{1+}\,\frac{x^{10}}{1+}\,\frac{x^{15}}{1+}\,\frac{x^{20}}{1+\ \cdots}$$

and

$$v = \frac{\sqrt[5]{x}}{1+}\,\frac{x}{1+}\,\frac{x^2}{1+}\,\frac{x^3}{1+\ \cdots}$$

then

$$v^5 = u\cdot\frac{1-2u+4u^2-3u^3+u^4}{1+3u+4u^2+2u^3+u^4}, \tag{1}$$

$$\frac{1}{1+}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+\ \cdots} = \left(\sqrt{\frac{5+\sqrt5}{2}}-\frac{\sqrt5+1}{2}\right)\sqrt[5]{e^{2\pi}} \tag{2}$$

and

$$\frac{1}{1-}\,\frac{e^{-pi}}{1-}\,\frac{e^{-2\pi}}{1-}\,\frac{e^{-3\pi}}{1-\ \cdots} = \left(\sqrt{\frac{5-\sqrt5}{2}}-\frac{\sqrt5-1}{2}\right)\sqrt[5]{e^{\pi}}, \tag{3}$$

while in his second letter he included the claim

$$\frac{1}{1+}\,\frac{e^{-2\pi\sqrt5}}{1+}\,\frac{e^{-4\pi\sqrt5}}{1+\ \cdots}$$
$$= \left[\frac{\sqrt5}{1+\sqrt[5]{\left\{5^{\frac34}\left(\frac{\sqrt5-1}{2}\right)^{\frac52}-1\right\}}}-\frac{\sqrt5+1}{2}\right]e^{2\pi/\sqrt5}. \tag{4}$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Twenty–seven years later, Hardy discusses the effect these identities had on him. "I should like you to begin by trying to reconstruct the immediate reaction of an ordinary professional mathematician who receives a letter like this from an unknown Indian clerk."

"The formulas (1), (2) and (4)" (Hardy got these a little confused here — (4) occurred only in the second letter) "are on a different level and obviously both difficult and deep. They defeated me completely. I had never seen anything in the least like them before. **A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them.**

Clearly, the three identities (1) — (3) had a great influence on Hardy's opinion of Ramanujan, the "unknown Indian clerk".

Two days later, Bertrand Russell writes to Lady Ottoline Morrell "In Hall I found Hardy and Littlewood in a state of wild excitement, because they believe they have discovered a second Newton."

Littlewood wrote "I can believe he's at least a Jacobi."

So I invite you to speculate on what might (or might not) have happened if Ramanujan had not included these three formulas in his first letter to Hardy.

You might even speculate that Ramanujan may never have gone to England (and where would you be now?).

In any case, what I intend to do in this talk is to go some way to proving formula (1). I hope that you might agree with me that formula (1) might well be "an identity that changed the course of history".

Jacobi's triple product identity

First, some notation.

$$(a;q)_n = \begin{cases} 1 & \text{if } n = 0 \\ (1-a)(1-aq)\ \cdots\ (1-aq^{n-1}) & \text{for } n \geq 1. \end{cases}$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ \cdots\ ,$$

$$(a_1, a_2,\ \cdots\ , a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty\ \cdots\ (a_n;q)_\infty,$$

$$\left(\begin{matrix} a_1, a_2, \cdots, a_m \\ b_1, b_2, \cdots, b_n \end{matrix}; q\right)_\infty = \frac{(a_1, a_2, \cdots, a_m; q)_\infty}{(b_1, b_2, \cdots, b_n; q)_\infty}.$$

Jacobi's triple product identity can be written (here $|q| < 1$ and $a \neq 0$),

$$(-a^{-1}q, -aq, q^2; q^2)_\infty = \sum_{k=-\infty}^{\infty} a^k q^{k^2}.$$

Proof of Jacobi's triple product identity

A simple proof might proceed as follows.

We can start with an identity equivalent to one of Euler's.

$$(-aq; q^2)_\infty = \sum_{k\geq 0} a^k \frac{q^{k^2}}{(q^2; q^2)_k}.$$

By induction on n,

$$(-a^{-1}q; q^2)_n(-aq; q^2)_\infty = \sum_{k=-n}^{\infty} a^k \frac{q^{k^2}}{(q^2; q^2)_{n+k}}.$$

Now let $n \to \infty$, and multiply by $(q^2; q^2)_\infty$ to obtain Jacobi's triple product identity.

Jacobi's triple product identity can be written in the equivalent form

$$(aq, a^{-1}q^2, q^3; q^3)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k a^k q^{(3k^2-k)/2}$$

of which the special case $a = 1$ is a celebrated identity of Euler,

$$(q; q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2-k)/2}.$$

It might be nice to have a combinatorial proof of Jacobi's triple product identity in this form, with powers of a attached to parts congruent to $+1$ modulo 3, and powers of a^{-1} attached to parts congruent to -1 modulo 3.

Also equivalent to Jacobi's triple product identity is

$$(a^{-1}, aq, q; q)_\infty = \sum_{k=0}^{\infty}(-1)^k (a^k - a^{-k-1}) q^{(k^2+k)/2}.$$

We can divide by $(1-a^{-1})$ (provided $a \neq 1$) and obtain

$$\begin{aligned}(a^{-1}q, aq, q; q)_\infty &= \sum_{k\geq 0}(-1)^k \frac{a^{k+\frac{1}{2}} - a^{-(k+\frac{1}{2})}}{a^{\frac{1}{2}} - a^{-\frac{1}{2}}}\, q^{(k^2+k)/2} \\ &= 1 + \sum_{k\geq 1}(-1)^k (a^k + a^{k-1} + \ \cdots\ + a^{-k}) q^{(k^2+k)/2}.\end{aligned}$$

If we let $a \to 1$ in this identity, we obtain a celebrated identity of Jacobi,

$$(q;q)_\infty^3 = \sum_{k\geq 0}(-1)^k (2k+1) q^{(k^2+k)/2}.$$

It might be noted that this can be written

$$(q;q)_\infty^3 = \sum_{k=-\infty}^{\infty}(4k+1) q^{2k^2+k},$$

and this leads directly to a proof of the two–squares theorem!

Most, but not all, of this section is in Hardy and Wright, but it will all appear in my book "The power of q".

An important special case of Jacobi's triple product identity

We have

$$(a^{-1}q, aq, q; q)_\infty = 1 + \sum_{k\geq 1}(-1)^k (a^k + a^{k-1} + \ \cdots\ + a^{-k}) q^{(k^2+k)/2}.$$

If we set $a = \eta = \exp \dfrac{2\pi i}{5}$, and use the facts that

$$\eta + \eta^{-1} = -\beta, \quad \eta^2 + \eta^{-2} = -\alpha,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$ (Fibonacci notation, not Ramanujan's) and

$$\eta^k + \eta^{k-1} + \cdots + \eta^{-k} = \begin{cases} 1 & \text{if } k = 5l,\ l \geq 0, \\ \alpha & \text{if } k = 5l+1,\ l \geq 0, \\ 0 & \text{if } k = 5l+2,\ l \geq 0, \\ -\alpha & \text{if } k = -5l-2,\ l \leq -1, \\ -1 & \text{if } k = -5l-1,\ l \leq -1, \end{cases}$$

we find that

$$\prod_{k\geq 1}(1+\beta q^k + q^{2k})(1-q^k) = (q^{10}, q^{15}, q^{25}; q^{25})_\infty - \alpha q(q^5, q^{20}, q^{25}; q^{25})_\infty.$$

If instead, we set $a = \eta^2$ (or simply change $\sqrt{5}$ to $-\sqrt{5}$), we find that

$$\prod_{k\geq 1}(1+\alpha q^k + q^{2k})(1-q^k) = (q^{10}, q^{15}, q^{25}; q^{25})_\infty - \beta q(q^5, q^{20}, q^{25}; q^{25})_\infty.$$

Obtaining Ramanujan's identity

Let

$$A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty.$$

Then we have

$$A - \alpha q B = \prod_{k\geq 1}(1+\beta q^k + q^{2k})(1-q^k)$$

and

$$A - \beta q B = \prod_{k\geq 1}(1+\alpha q^k + q^{2k})(1-q^k).$$

In other words, both $A - \alpha q B$ and $A - \beta q B$ are infinite products.

If we multiply these two, then divide by $(q^5; q^5)_\infty$, we obtain Ramanujan's (important) 5–dissection of Euler's product,

$$(q;q)_\infty = (q^{25}; q^{25})_\infty \left\{ \left(\begin{matrix} q^{10}, q^{15} \\ q^5, q^{20} \end{matrix} ; q^{25} \right)_\infty - q - q^2 \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix} ; q^{25} \right)_\infty \right\}.$$

This leads fairly easily to Ramanujan's "most beautiful identity"

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}$$

(see my 2011 Monthly paper) and to Ramanujan's congruences for $p(n)$ modulo powers of 5 (see my 1981 Crelle paper with David Hunt).

Now suppose we are inspired to consider the quantity

$$Q = \frac{(A-\alpha\eta qB)(A-\alpha\eta^{-1}qB)(A-\beta\eta^2 qB)(A-\beta\eta^{-2}qB)}{(A-\alpha\eta^2 qB)(A-\alpha\eta^{-2}qB)(A-\beta\eta qB)(A-\beta\eta^{-1}qB)}.$$

First, Q is an infinite product,

$$Q = \prod_{k\geq 1} t_k,$$

where

$$t_k = \frac{\begin{array}{c}(1+\alpha\eta^{2k}q^k+\eta^{4k}q^{2k})(1+\alpha\eta^{-2k}q^k+\eta^{-4k}q^{2k})\\ \times(1+\beta\eta^k q^k+\eta^{2k}q^{2k})(1+\beta\eta^{-k}q^k+\eta^{-2k}q^{2k})\end{array}}{\begin{array}{c}(1+\alpha\eta^k q^k+\eta^{2k}q^{2k})(1+\alpha\eta^{-k}q^k+\eta^{-2k}q^{2k})\\ \times(1+\beta\eta^{2k}q^k+\eta^{4k}q^{2k})(1+\beta\eta^{-2k}q^k+\eta^{-4k}q^{2k})\end{array}}$$

Now, if $k \equiv 0 \pmod 5$, $t_k = 1$.
If $k \equiv 1$ or $4 \pmod 5$,

$$t_k = \frac{\begin{array}{c}(1+\alpha\eta^2 q^k+\eta^4 q^{2k})(1+\alpha\eta^{-2}q^k+\eta^{-4}q^{2k})\\ \times(1+\beta\eta q^k+\eta^2 q^{2k})(1+\beta\eta^{-1}q^k+\eta^{-2}q^{2k})\end{array}}{\begin{array}{c}(1+\alpha\eta q^k+\eta^2 q^{2k})(1+\alpha\eta^{-1}q^k+\eta^{-2}q^{2k})\\ \times(1+\beta\eta^2 q^k+\eta^4 q^{2k})(1+\beta\eta^{-2}q^k+\eta^{-4}q^{2k})\end{array}}$$

$$= \frac{\begin{array}{c}(1-(1+\alpha)q^k+2\alpha q^{2k}-(1+\alpha)q^{3k}+q^{4k})\\ \times(1-(1+\beta)q^k+2\beta q^{2k}-(1+\beta)q^{3k}+q^{4k})\end{array}}{(1+q^k+q^{2k}+q^{3k}+q^{4k})(1+q^k+q^{2k}+q^{3k}+q^{4k})}$$

$$= \frac{(1-q^k)^2(1+\beta q^k+q^{2k})(1-q^k)^2(1+\alpha q^k+q^{2k})}{(1+q^k+q^{2k}+q^{3k}+q^{4k})^2}$$

$$= \frac{(1-q^k)^4(1+q^k+q^{2k}+q^{3k}+q^{4k})}{(1+q^k+q^{2k}+q^{3k}+q^{4k})^2}$$

$$= \frac{(1-q^k)^5}{1-q^{5k}}.$$

If $k \equiv 2$ or $3 \pmod 5$,

$$t_k = \frac{1-q^{5k}}{(1-q^k)^5}.$$

It follows that

$$Q = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)^5 \Big/ \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix}; q^{25}\right)_\infty .$$

On the other hand,

$$Q = \frac{(A-\alpha\eta qB)(A-\alpha\eta^{-1}qB)(A-\beta\eta^2 qB)(A-\beta\eta^{-2}qB)}{(A-\alpha\eta^2 qB)(A-\alpha\eta^{-2}qB)(A-\beta\eta qB)(A-\beta\eta^{-1}qB)}$$

$$= \frac{(A^2-qAB+\alpha^2q^2B^2)(A^2-qAB+\beta^2q^2B^2)}{(A^2+\alpha^2qAB+\alpha^2q^2B^2)(A^2+\beta^2qAB+\beta^2q^2B^2)}$$

$$= \frac{A^4-2qA^3B+4q^2A^2B^2-3q^3AB^3+q^4B^4}{A^4+3qA^3B+4q^2A^2B^2+2q^3AB^3+q^4B^4}.$$

Now suppose we know that

$$\frac{q}{1+}\,\frac{q^5}{1+}\,\frac{q^{10}}{1+}\,\frac{q^{15}}{1+}\,\frac{q^{20}}{1+\ \cdots} = q\left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix}; q^{25}\right)_\infty = q\,\frac{B}{A}$$

and that

$$\frac{\sqrt[5]{q}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ \cdots} = q^{\frac{1}{5}}\left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty .$$

If we write u for the first continued fraction, v for the second, then we have both

$$Q = \frac{v^5}{u}$$

and

$$Q = \frac{1 - 2u + 4u^2 - 3u^3 + u^4}{1 + 3u + 4u^2 + 2u^3 + u^4}.$$

(Ta–daa!)

Additional comments

The result that

$$\frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\ \cdots} = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty$$

is known as the Rogers–Ramanujan continued fraction. It is the special case $a = 1$ of

$$\frac{1}{1+}\,\frac{aq}{1+}\,\frac{aq^2}{1+}\,\frac{aq^3}{1+\ \cdots} = \frac{\displaystyle\sum_{k\geq 0}(-1)^k a^{2k} q^{(5k^2+3k)/2}(1 - aq^{2k+1})\frac{(aq;q)_k}{(q;q)_k}}{\displaystyle\sum_{k\geq 0}(-1)^k a^{2k} q^{(5k^2+k)/2}(1 - a^2q^{4k+2})\frac{(aq;q)_k}{(q;q)_k}}.$$

One can prove this result as follows. Let $G(a)$ denote the denominator of the right side. It can quite easily be shown (see my book) that

$$\frac{G(a)}{1 - aq} = G(aq) + aq(1 - aq^2)G(aq^2).$$

This can be written

$$\frac{G(a)}{(1 - aq)G(aq)} = 1 + aq \cdot \frac{(1 - aq^2)G(aq^2)}{G(aq)},$$

or,

$$\frac{(1 - aq)G(aq)}{G(a)} = \frac{1}{1 + aq \cdot \dfrac{(1 - aq^2)G(aq^2)}{G(aq)}}.$$

The result now follows by iteration.

Notice

My proof of formula (1) was inspired by Chadwick Gugg's 2009 paper in the Ramanujan Journal. Just a few days ago, I saw that Hei–Chi Chan had another proof in the Ramanujan Journal, in 2010.