My contact with Ramanujan

Michael D. Hirschhorn

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I first came across the Collected Papers in the library of UNSW about 50 years ago, when I was a schoolboy. I do not know how I first heard of Ramanujan. Was it possibly through Bell's "Men of Mathematics"? I have not checked whether the Collected Papers is still there, with my penned correction to the error, which has survived till today (!), in the first line of mathematics in Paper 29. What I do remember thinking, of identity (14) of Paper 25, "I will never understand this!" The identity is

$$\frac{(1-x^5)(1-x^{10})(1-x^{15})\ \cdots}{(1-x^{\frac{1}{5}})(1-x^{\frac{2}{5}})(1-x^{\frac{3}{5}})\ \cdots} = \frac{1}{\zeta^{-1} - x^{\frac{1}{5}} - \zeta x^{\frac{2}{5}}}$$

$$= \frac{\zeta^{-4} - 3x\zeta + x^{\frac{1}{5}}(\zeta^{-3} + 2x\zeta^2) + x^{\frac{2}{5}}(2\zeta^{-2} - x\zeta^3) + x^{\frac{3}{5}}(3\zeta^{-1} + x\zeta^4) + 5x^{\frac{4}{5}}}{\zeta^{-5} - 11x - x^2\zeta^5}$$

where

$$\zeta = \frac{(1-x)(1-x^4)(1-x^6)(1-x^9)\ \cdots}{(1-x^2)(1-x^3)(1-x^7)(1-x^8)\ \cdots}.$$

I see this now as the 5–dissection of Euler's product; I do understand it, and I have given two proofs of it (apart from the 'standard' proof using the quintuple product identity). I will mention one of these proofs later.

In 1969, I had eight months to occupy between finishing my first degree in Sydney and starting my second in Edinburgh. George Szekeres at UNSW was kind enough to take me on. At that time he was studying a problem concerning counting paths, and the generating functions involved the (infinite) Rogers–Ramanujan continued fraction. In trying to understand the fraction as a limit of the finite

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continued fraction, I was led to discover the formulas for the convergents,

$$\cfrac{1}{1+\cfrac{aq}{1+\ddots\ \cfrac{aq^n}{1}}} = \frac{P_{n-1}(aq)}{P_n(a)},$$

where

$$P_n(a) = \sum_{k\geq 0} a^k q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q .$$

and this was my first paper, accepted by Leonard Carlitz for the Duke Mathematical Journal. It turns out that the same formulas appeared in a paper by P. Kesava Menon in the Journal of the London Mathematical Society seven years earlier, and, as I discovered many years after, appear in Ramanujan's Notebooks (corrected by him!) So I was lucky to have this paper accepted. Was it because I had used Rogers's 19$^{\text{th}}$ Century notation $x_n! = (1-x)(1-x^2)\cdots(1-x^n)$?

In 1968 or early 1969, George Szekeres expanded the Rogers–Ramanujan continued fraction and its reciprocal (I don't know how far, but I can guess how he did it), and came to the conjecture that in both cases, the sign (+ or −) of the coefficients pretty early on became periodic with period 5. Using a crude form of the circle method, the "saddle point method" or "the method of steepest descents", Szekeres and I were able to calculate the dominant term, which involves a cosine term with period 5, but unfortunately were not able to prove the error small. This was done later by Szekeres and Bruce Richmond.

Indeed, if

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ \cdots}}}} = \sum_{n=0}^{\infty} u_n q^n = \frac{(q,q^4,q^5;q^5)_\infty}{(q^2,q^3,q^5;q^5)_\infty}$$

and

$$1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\ \cdots}}}x = \sum_{n=0}^{\infty} v_n q^n = \frac{(q^2,q^3,q^5;q^5)_\infty}{(q,q^4,q^5;q^5)_\infty},$$

Richmond and Szekeres showed that

$$u_n \sim \frac{\sqrt{2}}{(5n)^{\frac{3}{4}}} \exp\left\{\frac{4\pi}{25}\sqrt{5n}\right\}\left(\cos\left(\frac{4\pi n}{5}+\frac{3\pi}{25}\right)+\mathrm{O}(n^{-\frac{1}{2}})\right)$$

and

$$v_n \sim \frac{\sqrt{2}}{(5n)^{\frac{3}{4}}} \exp\left\{\frac{4\pi}{25}\sqrt{5n}\right\}\left(\cos\left(\frac{2\pi n}{5}-\frac{4\pi}{25}\right)+\mathrm{O}(n^{-\frac{1}{2}})\right).$$

With the benefit of MAPLE, as I recently discovered, it is easy and instructive to compare these formulas with reality.

By interpreting combinatorially some formulas in the Lost Notebook, George E. Andrews was able to confirm Szekeres's original observation, and some time later, I showed, using the quintuple product, that

$$\sum_{n=0}^{\infty} u_{5n}q^n = 1/(q,q^2,q^3,q^3,q^4,q^5,q^7,q^8,q^9,q^{10},q^{11},q^{12},q^{13},$$
$$q^{14},q^{15},q^{16},q^{17},q^{18},q^{20},q^{21},q^{22},q^{22},q^{23},q^{24};q^{25})_\infty,$$

with similar formulas for all the other nine series, which also prove Szekeres's observations.

What came as a surprise is that in the Lost Notebook, in a table on page 49, we find that Ramanujan had (correctly!) calculated the u_n and v_n beyond 1000. But he left us no conclusions!

George E. Andrews spent the (northern) academic year 1978–79 in Australia, during which time he supervised my Ph.D. He drew my attention to a two–line statement on page 41 of the Lost Notebook,

$$\text{If}\quad G(a,\lambda) = \sum_{k=0}^{\infty} \frac{q^{(k^2+k)/2}(a+\lambda)\cdots(a+\lambda q^{k-1})}{(1-q)\cdots(1-q^n)(1+bq)\cdots(1+bq^n)}$$

$$\text{then}\qquad \frac{G(aq,\lambda q)}{G(a,\lambda)} = \cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{1+\cfrac{bq^2+\lambda q^4}{1+\ \cdots}}}}}.$$

He told me he had proved this result "with some difficulty", and challenged me to find the convergents.

A few weeks later, he came in to find a note on his office door "I have found a formula for the convergents, but this note is too small to contain it."

$$\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\ddots\cfrac{aq^n+\lambda q^{2n-1}}{1}}}} = \frac{P_{2n-2}(b,aq,\lambda q)}{P_{2n-1}(a,b,\lambda)},$$

$$\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\ddots\cfrac{bq^n+\lambda q^{2n}}{1}}}} = \frac{P_{2n-1}(b,aq,\lambda q)}{P_{2n}(a,b,\lambda)},$$

where

$$P_n(a,b,\lambda) = \sum_{s,t,u\ge 0} a^s b^t \lambda^u q^{(s+t)(s+t+1)/2+(s+t)u+u^2}$$
$$\times \begin{bmatrix} n+1-s-t-u \\ u \end{bmatrix}_q \begin{bmatrix} \lfloor (n+1)/2 \rfloor - t - u \\ s \end{bmatrix}_q \begin{bmatrix} \lfloor n/2 \rfloor - s - u \\ t \end{bmatrix}_q .$$

with the special convention $\begin{bmatrix} -1 \\ 0 \end{bmatrix}_q = 1$.

In 1980, just after I graduated, I saw how the facts that

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

$$\frac{1}{(q;q)_\infty} = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6} \{R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1}$$
$$+5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\}$$

and

$$R(q)^{-5} - 11q - q^2R(q)^5 = \frac{(q;q)_\infty^6}{(q^5;q^5)_\infty^6}$$

where

$$R(q) = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix} ; q \right)_\infty$$

yield

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^6}{(q;q)_\infty^6},$$

and indeed yield formulas for $\sum_{n=0}^{\infty} p(5^\alpha n + \delta_\alpha)q^n$ for all $\alpha \geq 1$,

thus yielding proofs of Ramanujan's partition congruences for powers of 5.

You may be interested to hear that I calculated the formula for $\sum_{n=0}^{\infty} p(25n+24)q^n$ by hand, and went some way to calculating the formula for $\sum_{n=0}^{\infty} p(125n+99)q^n$ before enlisting the help of David Hunt and his computer.

When Frank Garvan came to UNSW in 1981, he soon gave the corresponding proof for powers of 7. Frank went on to do his Doctorate with George, and as you are aware, has gone on to do great things.

Our proofs involve the use of matrices to state the recurrence for certain coefficient vectors. It is interesting to me that Ramanujan appears never to have used matrices (?)

For his 100^{th} Birthday, I presented Ramanujan with a proof that $p(11n+6) \equiv 0 \pmod{11}$. My proof consisted of the statement of a formula that entirely filled a page of the Monthly. I realised later on that I had inadvertently reproduced some earlier work of Frank's.

The editor of the Monthly printed the paper in such a way that turning the page revealed the entire formula in all its terrifying beauty.

In a seminal paper, Frank Garvan and Dennis Stanton gave completely new formulas for $\sum_{n=0}^{\infty} p(5n+4)q^n$, $\sum_{n=0}^{\infty} p(7n+5)q^n$ and $\sum_{n=0}^{\infty} p(11n+6)q^n$ which enabled them use affine maps to prove the three corresponding congruences. I completed squares in their formulas, and this gave slightly simpler proofs using linear maps. However, if I am right, their affine maps led them, if not to create, then at least to reinforce the idea of a crank.

At the Ramanujabn Centenary conference in Illinois, Bill Gosper gave a plenary address, “Ramanujan as Nemesis”, and I made a regrettable error.

Bill started by saying, and this is often badly misquoted,

“How can we pretend to love this guy [Ramanujan], when he is forever reaching out from the grave to snatch away our neatest results?”

Bill then proceeded to state a conjecture which he said “Even Erdös couldn’t do.”.

The conjecture was

“If a number is the sum of four distinct odd squares then it is the sum of four distinct even squares.”

(The converse is false. A number which is congruent to 0 modulo 8 may be the sum of four distinct even squares, but certainly isn’t the sum of four odd squares. The converse does hold for numbers congruent to 4 modulo 8.)

The mistake I made was to solve this problem and to show Bill my solution at the end of his talk. The consequence was that the published version of his talk did not contain the conjecture, and my proof has never been published.

Here it is. I leave it to you to provide the details.

$$\begin{aligned}(4a+1)^2 &+ (4b+1)^2 + (4c+1)^2 + (4d+1)^2 \\ &= (2a+2b+2c+2d+2)^2 \\ &+ (2a-2b-2c+2d)^2 \\ &+ (2a+2b-2c-2d)^2 \\ &+ (2a-2b+2c-2d)^2.\end{aligned}$$

Bill’s conjecture led me to further discoveries. For a number $n \equiv 4 \pmod 8$, the number of partitions of n into four distinct odd squares is at least as great as the number of partitions of n into four distinct even squares, but never more than twice as great! I leave it to you to ponder this problem. (I <u>do</u> know why!)

To solve the above problem, I used a technique involving generating functions that I developed in conjunction with my friend and collaborator James Sellers. We have made many discoveries, among them the following.

If we let $p_{4\square}(n)$, $p^{+}_{4\square}(n)$, $p^{d}_{4\square}(n)$ and $p^{d+}_{4\square}(n)$ denote, respectively, the number of partitions of n into four squares, four positive squares, four distinct squares and four distinct positive squares, then, if $n \equiv 69 \pmod{72}$, all four functions of n are even. We gave a combinatorial proof of these results.

Together with Shaun Cooper, we found generating function formulas that provide instant proofs of these results. These results were supposed to be published in the Proceedings of a conference held in Mysore by the Jangjeon Mathematical Society, but I have never seen a copy! So these results are probably new to you.

$$\begin{aligned}
\sum_{n=0}^{\infty} p_{4\Box}(72n+69)q^n &= 2\left(\Pi_1+\Pi_2+\Pi_4\right), \\
\sum_{n=0}^{\infty} p_{4\Box}^{+}(72n+69)q^n &= 2\left(\Pi_1+\Pi_3+\Pi_4\right), \\
\sum_{n=0}^{\infty} p_{4\Box}^{d}(72n+69)q^n &= 2\left(\Pi_1-\Pi_3+\Pi_4\right), \\
\sum_{n=0}^{\infty} p_{4\Box}^{d+}(72n+69)q^n &= 2\left(\Pi_1-\Pi_2+\Pi_4\right),
\end{aligned}$$

where

$$\begin{aligned}
\Pi_1 &= \left(\begin{matrix} q^6,q^6,q^{10},q^{12},q^{12},q^{14},q^{18},q^{18},q^{24},q^{24},q^{24} \\ q,q,q^5,q^7,q^{11},q^{11},q^{13},q^{13},q^{17},q^{19},q^{23},q^{23} \end{matrix};q^{24}\right)_\infty, \\
\Pi_2 &= \left(\begin{matrix} q^4,q^8,q^{12},q^{16},q^{20},q^{24},q^{24},q^{24} \\ q,q^2,q^7,q^{10},q^{14},q^{17},q^{22},q^{23} \end{matrix};q^{24}\right)_\infty, \\
\Pi_3 &= q^2\left(\begin{matrix} q^4,q^8,q^{12},q^{16},q^{20},q^{24},q^{24},q^{24} \\ q^2,q^5,q^{10},q^{11},q^{13},q^{14},q^{19},q^{22} \end{matrix};q^{24}\right)_\infty, \\
\Pi_4 &= q^3\left(\begin{matrix} q^2,q^6,q^6,q^{12},q^{12},q^{18},q^{18},q^{22},q^{24},q^{24},q^{24} \\ q,q^5,q^5,q^7,q^7,q^{11},q^{13},q^{17},q^{17},q^{19},q^{19},q^{23} \end{matrix};q^{24}\right)_\infty.
\end{aligned}$$

I have also attempted to apply the crude circle method I learnt from Szekeres to the problem of the number of partitions of a number into (any number of) distnct squares, and found that

$$p_{\Box}^{d}(n) \sim \frac{\Lambda^{\frac{1}{3}}}{\sqrt{6\pi}n^{\frac{5}{6}}} \exp\left\{3\Lambda^{\frac{2}{3}}n^{\frac{1}{3}}\right\}$$

where

$$\Lambda = \int_0^{\infty} \frac{u^2e^{-u^2}}{1+e^{-u^2}}\,du.$$

Agreement is pretty good (less than 10% error) at arond $n = 2000$.

But again, I cannot show the error is small.

Returning to Ramanujan's mathematics, my colleague Shaun Cooper recently discovered that certain identities of Ramanujan's can be factorised. That is, we can find a pair of identities which when multiplied together give the identity in question. These results of ours were published in Acta Mathematica Sinica, and I just gave a talk on them in Mysore.

If we set

$$A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty$$

then it is an easy consequence of Jacobi's triple product identity that

$$A - \alpha q B = \prod_{k=1}^{\infty}(1 + \beta q^k + q^{2k})(1 - q^k),$$
$$A - \beta q B = \prod_{k=1}^{\infty}(1 + \alpha q^k + q^{2k})(1 - q^k),$$

and if we multiply these together, we obtain

$$A^2 - qAB - q^2B^2 = (q;q)_\infty (q^5;q^5)_\infty.$$

trivially equivalent to Ramanujan's 5–dissection of Euler's product,

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left\{ R(q^5)^{-1} - q - q^2 R(q^5) \right\}.$$

This is the proof I promised early on in this talk.

Factorisations of Ramanujan's identities remain an interest of mine

Over the years I have investigated many other statements of Ramanujan, for example in diophantine equations. I believe I know how Ramanujan proceeded from the identity

$$(3a^2 + 5ab - 5b^2)^3 + (4a^2 - 4ab + 6b^2)^3 + (5a^2 - 5ab - 3b^2)^3 = (6a^2 - 4ab + 4b^2)^3$$

to the statement

$$
\begin{aligned}
\text{if} \qquad & \sum_{n=0}^{\infty} a_n x^n = \frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3}, \\
& \sum_{n=0}^{\infty} b_n x^n = \frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3}, \\
& \sum_{n=0}^{\infty} c_n x^n = \frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3}, \\
\text{then} \qquad & a_n^3 + b_n^3 = c_n^3 + (-1)^n
\end{aligned}
$$

but I don't know how Ramanujan found the above formula, nor why he may have suspected such a formula exists.

(Incidentally, I discovered how he proceeded by working backwards. I was inspired to change the 82's to 2's, and found squares (of Fibonacci numbers) play a role.)

I have investigated statements made by Ramanujan in elementary algebra. For example,

$$
\begin{aligned}
&\left\{-c + \sqrt{(b+c)(c+a)}\right\} \left\{-a + \sqrt{(c+a)(a+b)}\right\} \left\{-b + \sqrt{((a+b)(b+c)}\right\} \\
&\qquad = 2 \left(\frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{ab + bc + ca} \right)^{-2}.
\end{aligned}
$$

Proof: Let

$$
\sqrt{(c+a)(a+b)} = X, \quad \sqrt{(a+b)(b+c)} = Y, \quad \sqrt{(b+c)(c+a)} = Z.
$$

Then

$$
X^2 - a^2 = Y^2 - b^2 = Z^2 - c^2 = ab + bc + ca,
$$

and it is easy to check that

$$
(X+a)(Y+b)(Z+c) = (ab+bc+ca)(X+Y+Z+a+b+c).
$$

It is then easy to compute

$$
(X-a)(Y-b)(Z-c),
$$

and obtain Ramanujan's result. (This is joint work with Vasile Sinescu.)

Ramanujan gave his own version of Stirling's formula,

$$\Gamma(x+1) = \sqrt{\pi}\left(\frac{x}{e}\right)^x \left(8x^3 + 4x^2 + x + \frac{\theta(x)}{30}\right)^{\frac{1}{6}}.$$

He claimed that $\frac{3}{10} < \theta(x) < 1$ and that $\theta(x) \to 1$ as $x \to \infty$. This claim was proved recently by Ekatherina Karatsuba. Her proof is necessarily fairly technical. However, if we restrict ourselves to integers n, I showed, using only the series for $\log(1+x)$, that

$$1 - \frac{11}{8n} + \frac{79}{112n^2} < \theta(n) < 1 - \frac{11}{8n} + \frac{79}{112n^2} + \frac{20}{33n^3},$$

and hence that $\theta(n)$ is increasing, and that $\theta(n)$ is concave, that is, the second difference is negative. (This is joint work with Mark Bertram Villarino.)

All this is well illustrated by graphs in MAPLE.

My final example is a formula (described by Bruce Berndt as enigmatic) for the harmonic series given by Ramanujan,

$$\sum_{k=1}^{n} \frac{1}{k} = \gamma + \frac{1}{2}\log 2m + \frac{1}{12m} - \frac{1}{120m^2} + \frac{1}{630m^3} - \cdots$$

where

$$m = \frac{n(n+1)}{2}.$$

(Ramanujan gave 9 terms, the last of which is $\frac{140051}{17459442m^9}$.

Using the simple technique of telescoping series, I was able to give a straightforward explanation.

My work on Ramanujan has led to collaboration with around 40 people so far, to whom I am forever grateful, and I am sure will lead to more.

I have written almost exactly 100 papers on Ramanujan's work (so far), all of them available on my personal web–page at UNSW, accessible via Google.

Thank you,

and, Happy Birthday, Ramanujan!