The University of New South Wales
School of Mathematics and Statistics

Mathematics Drop–in Centre

LOGARITHMS

"Logarithm" is really just another word for "exponent" or "power". In an expression a^b we call a the base and b the logarithm. So if

$$a^b = c$$

then we say that b is the logarithm of c to the base a, written

$$b = \log_a c\ .$$

In all of these equations, a and c must be positive numbers; b may be positive, negative or zero. In principle, logarithms can be evaluated by rewriting them as powers.

Example. Evaluate $\log_2 32$.
Solution. Let $x = \log_2 32$. Then $2^x = 32$ and so by trial and error $x = 5$.

In practice, evaluating logarithms usually requires a calculator. For instance, if $x = \log_2 34$ then $2^x = 34$; the previous example shows that x must be a bit more than 5 and definitely less than 6, but it's difficult to pin it down much more closely than this.

Logarithm laws. The following identities hold, where a, x, y are positive numbers and p is any real number:

$$\begin{aligned} \log_a(xy) &= \log_a x + \log_a y \\ \log_a\left(\frac{x}{y}\right) &= \log_a x - \log_a y \qquad (*) \\ \log_a(x^p) &= p\log_a x\ . \end{aligned}$$

In these equations it does not matter what base we use for the logarithms, so long as it is the same throughout the equation. So we shall often write simply $\log(xy) = \log x + \log y$ and so on. Combining the above properties we have, for example,

$$\begin{aligned} &7\log(x^2y^5) - 9\log(xy^4) \\ &\qquad = 7(2\log x + 5\log y) - 9(\log x + 4\log y) \\ &\qquad = 5\log x - \log y \\ &\qquad = \log\left(\frac{x^5}{y}\right)\ . \end{aligned}$$

Note that there is **no useful simplification** for $\log(x \pm y)$.

Further identities follow from the definition of the logarithm:

$$a^{\log_a x} = x \quad \text{and} \quad \log_a(a^x) = x\ .$$

In this case the base of the logarithm *does* matter and cannot be omitted. A sample simplification:

$$100^{\log_{10} x} = \left(10^{\log_{10} x}\right)^2 = x^2\ .$$

An especially important logarithm is the one with base $e = 2{\cdot}71828\cdots$. It is called the **natural logarithm** and written ln. That is, $\ln x$ means the same as $\log_e x$. All of the identities $(*)$ remain true when $\log_a$ is replaced by ln. For example,

$$\ln\left(\frac{e^3}{\sqrt{x}}\right) = \ln(e^3) - \ln(x^{1/2}) = 3\ln e - \tfrac{1}{2}\ln x = 3 - \tfrac{1}{2}\ln x\ .$$

Comment. We have ignored certain difficulties in defining logarithms, and a more careful approach will be taken in MATH1131 lectures. Nevertheless, the properties given will remain true and we can still manipulate expressions as in the previous examples.

EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop–in Centre.

1. Write the following statements as equations involving logarithms:

   (a) $1331 = 11^3$; (b) $25^{1/2} = 5$; (c) $y = x^{\pi}$;

   (d) $p^q = 7$; (e) $2^x = \frac{3}{2}$; (f) $z^{-1\cdot 23} = 4\cdot 56$.

2. Write the following as equations involving powers:

   (a) $7 = \log_3 2187$; (b) $x = \log_{10} 2$; (c) $\log_5 a = -3$;

   (d) $x = \ln 5$; (e) $\ln x = 5$.

3. Evaluate, without using a calculator,

   (a) $\log_2 128$; (b) $\log_{125} 5$; (c) $\log_{10} \frac{1}{100}$.

4. Simplify if possible

   (a) $\log_2\left(\frac{x^2}{y^5}\right) + \log_2\left(\frac{8y^7}{x^9}\right)$; (b) $3^{\log_3 x - 2\log_3 y}$;

   (c) $\log(x^2 + y^2)$; (d) $4\ln\left(\frac{s^5}{t^6}\right) + \ln\left(\frac{s^7}{t^8}\right)$;

   (e) $\log_{18}(2^z 3^{2z})$; (f) $(\ln a)(\ln b)$;

   (g) $\ln 10 + \ln 100 + \ln 1000$; (h) $\ln\big((xy)^{-3}\big) - \ln\left(\frac{x^6}{y^7}\right)$.

ANSWERS.

1. (a) $\log_{11} 1331 = 3$;

   (b) $\log_{25} 5 = \frac{1}{2}$;

   (c) $\log_x y = \pi$;

   (d) $\log_p 7 = q$;

   (e) $x = \log_2 \frac{3}{2}$;

   (f) $\log_z 4\cdot 56 = -1\cdot 23$.

2. (a) $3^7 = 2187$;

   (b) $10^x = 2$;

   (c) $a = 5^{-3}$;

   (d) $e^x = 5$;

   (e) $x = e^5$.

3. (a) 7; (b) $\frac{1}{3}$; (c) -2.

4. (a) $3 - 7\log_2 x + 2\log_2 y$, or $3 + \log_2\left(\frac{y^2}{x^7}\right)$;

   (b) $\frac{x}{y^2}$;

   (c) no simplification;

   (d) $27\ln s - 32\ln t$, or $\ln\left(\frac{s^{27}}{t^{32}}\right)$;

   (e) z;

   (f) no simplification; the given expression can be written as $\ln(b^{\ln a})$ or $\ln(a^{\ln b})$, but these are not really simpler;

   (g) $6\ln 10$;

   (h) $-9\ln x + 4\ln y$, or $\ln\left(\frac{y^4}{x^9}\right)$.