National Curriculum issues and opportunities for revitalising geometric thinking in the classroom

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August 2012

Overview

Today's talk:

- A) Drafts for Mathematical Methods and Specialist Mathematics (ACARA) for Advanced Mathematics in Years 11,12

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- C) The role of geometry in maths education and technology

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- A) Drafts for Mathematical Methods and Specialist Mathematics (ACARA) for Advanced Mathematics in Years 11,12
- B) Response from School of Mathematics and Statistics UNSW
- C) The role of geometry in maths education and technology
- D) A brief intro to some interesting possible topics for high school: *Cartesian geometry and conics, the rational parametrization of a circle, Pappus and Pascal's theorems in projective geometry, triangle centers and the Euler line, Archimedes law of the lever and convex combinations, centers of mass and means of probability distributions, Olympic rankings and rational trigonometry*!

Mathematical Methods (MM)

ACARA has released drafts for two advanced level Year 11 and 12 mathematics courses: *Mathematical Methods (MM)* and *Specialist Mathematics (SM)*, roughly at the level of two-unit and four-unit. Here are the proposed topics for the four units of MM:

UNIT 1: Algebra, functions and graphs 1, Calculus 1, Probability

UNIT 2: Algebra, functions and graphs 2, Calculus 2, Discrete Random Variables

UNIT 3: Calculus 3, Calculus 4, Continuous Random Variables

UNIT 4: Interval estimates for proportions and means, Calculus 5

[http://www.acara.edu.au/verve/_resources/Mathematical+Methods.pdf]

Specialist Mathematics (SM)

Here are the proposed topics for the four units of SM:

UNIT 1: Recurrence Relations, Combinatorics, Geometry, Vectors in the plane

UNIT 2: Trigonometry, Matrices, Real and Complex Numbers, Graph theory

UNIT 3: Vectors in three dimensions, Matrices and Systems of Equations, Complex numbers, Functions and Calculus

UNIT 4: Further Calculus and Applications of Calculus, Statistical Inference by Continuous Data

[http://www.acara.edu.au/verve/_resources/Specialist+Mathematics.pdf]

UNSW and G08 response

In March, ACARA presented the current drafts to the UNSW School of Mathematics and Statistics, *to considerable concern*. A working party was formed to study these drafts and form a response.

Four authors contributed to the report: **Assoc. Prof. Norman Wildberger** (Chair), **Mr. Peter Brown** (Director of First Year), **Dr. Daniel Chan** (Dept of Pure Mathematics), and **Assoc. Prof. David Warton** (Dept of Statistics).

You can view the report at *http://www.maths.unsw.edu.au/* under NEWS.

Our report is highly critical of the ACARA drafts: **The proposed national curriculum will be, in our opinion, a setback for mathematics education in NSW, and we would support the Federal government having a fresh look at the project.**
This report now has the backing of 5 of the Go8 universities.

Response to ACARA drafts

Specifically, our main concerns are that the current draft has

a) unbalanced syllabus content out of line with top international standards
b) emphasis on rote learning rather than mathematical understanding, with a notable absence of engaging applications
c) unclear design methodology, mathematical inconsistencies, and lack of justification
d) excessive implementation costs for teachers and State Boards of Education.

In particular, *NSW will miss out on the middle level 3-unit course*. Engineering and science students will generally not take SM, and *will not get adequate preparation for their tertiary studies from MM.*

Unbalanced syllabus content out of line with top international standards

The draft national curriculum replaces core material of *algebra*, *geometry* and *applications of calculus* with *a lot of advanced statistics* and, for the higher strand, *tertiary level topics*. The subject MM is now *largely calculus and statistics*. The subject SM is an overfull compilation of advanced topics, which will only appeal to a small cohort of students.

The approach to Statistics in MM is a standard one but at too high a level, adapted for students without the requisite foundations in calculus, requiring so many corners to be cut that the mathematical underpinnings of the subject become lost, and the subject becomes largely a 'black-box 'exercise.

Lopsided Mathematical Methods (MM)

The MM syllabus especially is seriously inadequate, as students will get only a minimum of algebra, with hardly any review, and miss out entirely on any kind of geometry beyond that in the year 10 syllabus, with hardly any important applications of calculus. This lack of balance is not in-line with what our competitors in Asian countries are doing.

Geometric thinking is absolutely vital to much of science and most of engineering, and many students appreciate the strongly visual aspect of this subject, as opposed to the formula-driven approach to calculus currently in the draft. Geometric intuition, which includes coordinate geometry, vector geometry, three dimensional geometry as well as classical Euclidean geometry, is best built up steadily.

Rote learning rather than conceptual understanding

The MM syllabus is heavily symbolic and rule-driven, with a lack of clear unifying ideas and themes. It focuses more on accumulating methodological tools rather than on developing mathematical understanding; teaching students how to compute answers without teaching why methods work.

Interesting applications of Calculus—to motion, Newton's laws, volumes etc—are largely missing. In MM, the main application of Calculus seems to be Statistics!

Students ought to see topics that are *interesting* as well as *useful*, link with and reinforce each other, and which respect the historical development of the subject.

And there should be *gems, sparkle*, and *challenges*!

Unclear design, inconsistencies, lack of justification

The Draft documents are put together sloppily, with many mathematical inconsistencies, a singular lack of explanation and justification, and liberal use of cut and pastes. [For details, see our report.] Where are the scientific studies that back up the radical proposals for redistribution of the mathematical landscape being proposed here?

For two thousand years, mathematics was a balance between geometry, arithmetic/algebra, and then analysis. The reality is that

$$Mathematics \neq Calculus + Statistics$$

The current draft for MM is a distortion of the historical reality and the pervailing international consensus. Where is the studies and trials that support the underlying assumptions? Without these, the exercise runs the danger of becoming yet one more *unsubstantiated mathematical fad.* Dangerously, one that is poised to *weaken* mathematics education in Australia.

Costs to teachers

The new Statistics section is going to be *very painful* to implement. There is no breakdown of topics into smaller lesson plans, or detailed provision of examples, representative exercises to help ease the pain.
Each State's BOS will have to reinvent the wheel trying to patch the deficiencies. The current NSW curriculum in mathematics is generally excellent, and of international quality; it ought to be the benchmark by which others are judged in Australia. Abandoning it in favour of a radical, untested and lopsided new curriculum is hardly a wise move. *Another case of politics trumping common sense?*

Any major rewrite of the curriculum ought to be accompanied by a lot more information and resources. Asian countries use *textbooks*, and the AMSI books for years 1-10 are a great addition, but what is there to support the new proposed curricula in Years 11 and 12?

The UK experience after 1988

After the 1988 Education Act, England and Wales moved to a National Curriculum for the 5-16 age range. While geometry still featured, the cognitive demands on students became: "know", "identify", "recognise", "use", "understand", "determine" and "calculate"; rather than "*explain*" and "*prove*". In 1995, "*inadequate attention to precision and proof in school mathematics and consequent difficulties in transition to higher education were identified as serious parts of the problem*[1]" in a report from the London Mathematical Society called "Tackling the mathematics problem".

What went wrong??

1. Michael Price, A Century of School Geometry Teaching: From Euclid to the "Subject Which Dare Not Speak Its Name"? in *The Changing Face of Geometry: Celebrating a Century of Geometry and Geometry Teaching*, Ed. Chris Pritchard, Cambridge Univ Press, 2003.

A similiar story??

Margaret Brown: 1991 MA Presidential address:

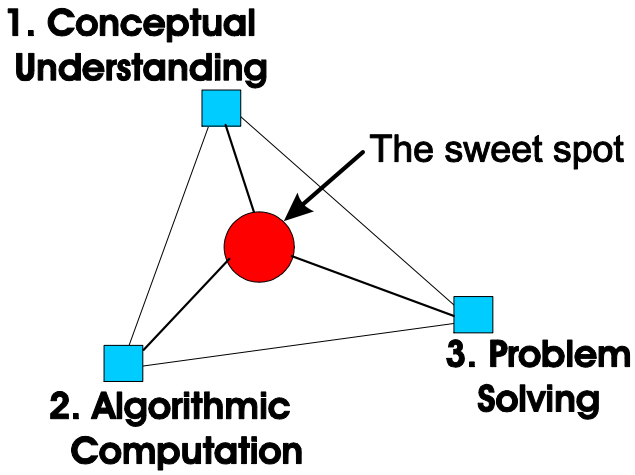
"the haste with which the subject groups (especially those in mathematics and science) had to report, and the management structure chosen, prevented any serious collection or discussion of essential information, such as comparative information from other countries, proper theoretical underpinnings of a curriculum framework, or research results; consultation was not taken seriously, either in the way it was conducted or in the regard for the results; essential trialling and piloting of the attainment targets and programmes of study did not take place, and the trialling of the assessment was disrupted by policy changes; the implementation in schools was premature, before teachers had had sufficient information, teaching resources, specialized in-service input, or planning time[2]."

2. M. Brown, The second iteration, *Math.Gaz.* 75 (October 1991), 263-274.

Three spokes of mathematics teaching

Effective mathematics learning is a balance of

1. **conceptual understanding**



Many teachers/institutions are *oriented to the second spoke:* **algorithmic computation**, since it is relatively easy to teach (if you don't mind lots of uninterested students!), is the easiest to assess, and presents a small target.
But for long-term learning, all three aspects ought to be represented. Curriculum development should also keep these in balance. In particular, the third spoke must be represented: a *serious curriculum comes with a list of representative sample problems*.

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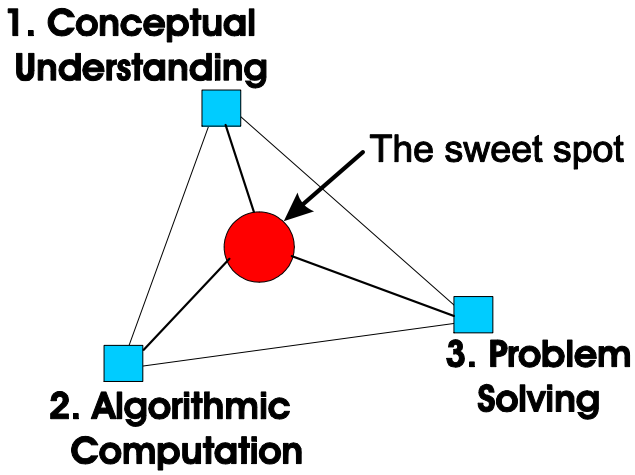

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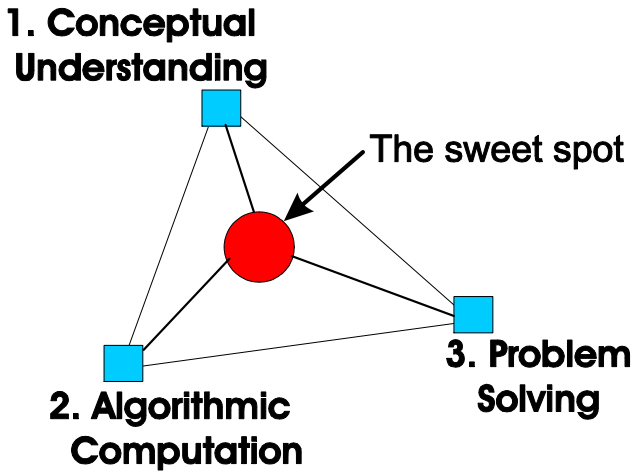

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Rethinking the role of geometry

The time is right for a rethink about the role of geometry. As opposed to ACARA, I firmly believe that we need to *increase* the amount and quality of geometry in school learning, both in primary and high schools, and across all levels.

- Many students are more visually oriented rather than algebraically oriented

The rest of the talk gives a kaleidoscopic introduction to some geometry topics that are accessible to the advanced high school maths student, are historically important, and interesting!

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- geometry is the best topic for introducing conceptual understanding and problem solving
- geometry is the subject which most obviously has *theorems*, so that the role of proofs can be discussed, and for stronger students, tackled.

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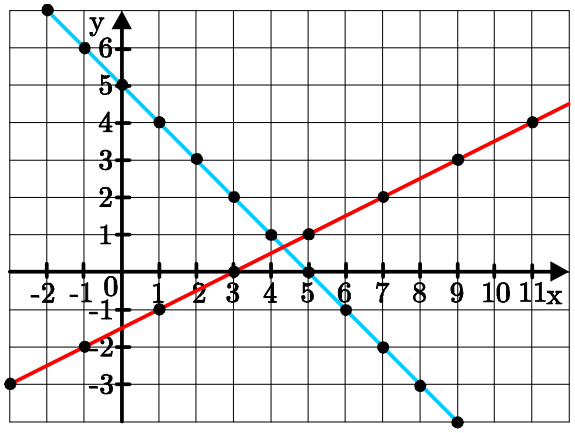
Cartesian plane and equations

Around 1600 Descartes introduced coordinates into geometry. This allowed algebra and geometry to come together: one of the truly great developments of modern mathematics. Here for example are the lines $x + y - 5 = 0$ (blue) and $x - 2y - 3 = 0$ (red) and they meet in the point $\left[\frac{13}{3}, \frac{2}{3}\right]$.

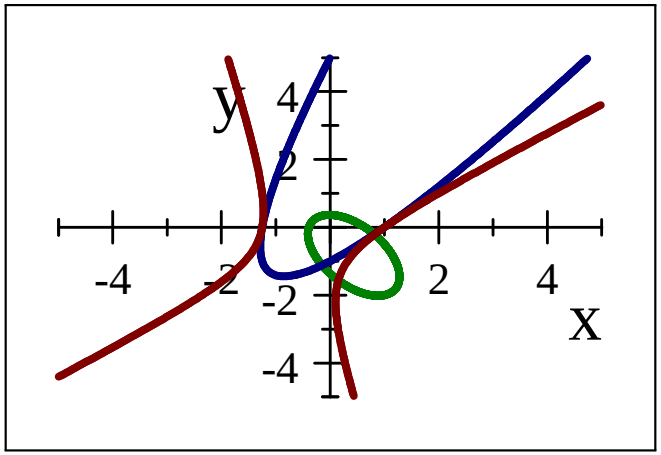


It is unfortunate if high school students don't get much beyond this very simple set-up. The 17th century mathematicians went a lot further (Descartes, Fermat, Newton, Bernoullis).

Second degree equations

Descartes realized that *second degree algebraic relations* between x and y correspond exactly to the *conic sections* of the Greeks: *ellipses, parabolas* and *hyperbolas*. For example the following equations give one of each.

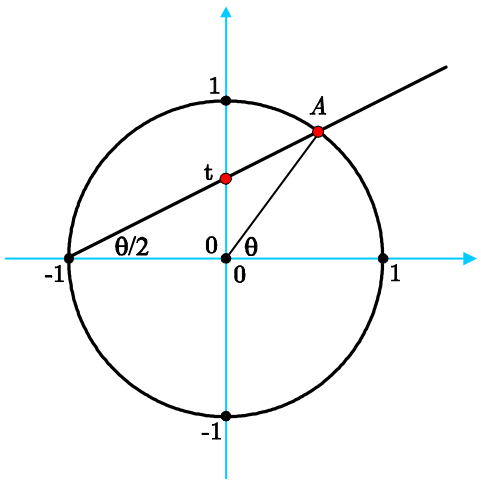
$$\begin{aligned} 4x^2 + 3xy + 2y^2 - x + 2y &= 1 \\ 4x^2 - 4xy + y^2 + x - 4y &= 5 \\ 4x^2 - 4xy - y^2 + x - 4y &= 5 \end{aligned}$$



Intriguing question: how do you tell, from the equation, which is which? This is a key application of *linear algebra*!

Rational parametrization of the unit circle

The unit circle
$x^2 + y^2 = 1$ has a *rational parametrization*, going back essentially to Pythagoras, or at least to Euclid:



$$A = \left[\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right] \qquad t \in (-\infty, \infty)$$

High school students *should* learn this as
a more accurate parametrization than $[\cos\theta, \sin\theta]$.

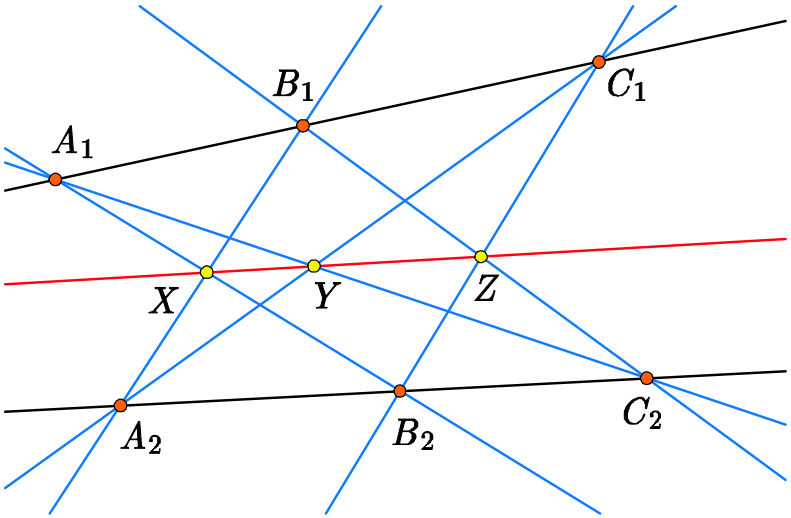
It also leads to the rationalizing substitution

$$\tan\frac{\theta}{2} = t \qquad \cos\theta = \frac{1-t^2}{1+t^2} \qquad \sin\theta = \frac{2t}{1+t^2}$$

to simplify integrals with circular functions.

Pappus' theorem and projective geometry

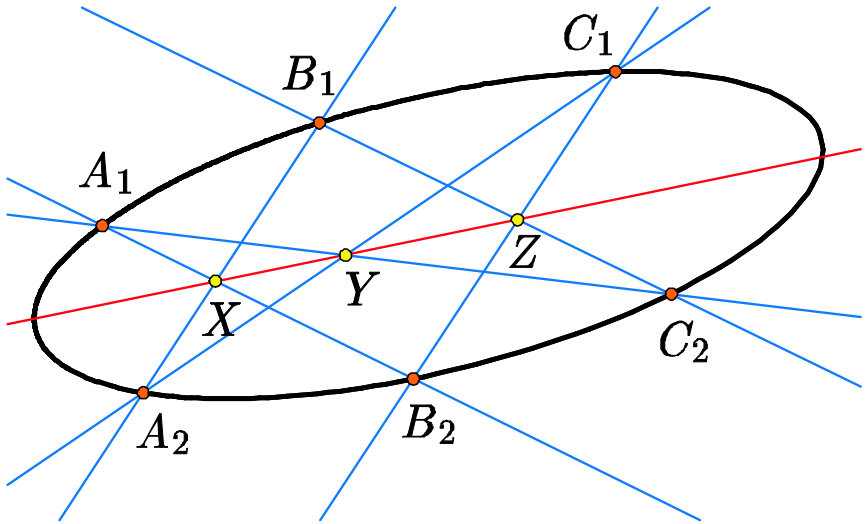
Start with A_1, B_1, C_1 collinear and A_2, B_2, C_2 collinear. *Then* X, Y, Z *are also collinear.* (Pappus 300 A.D.) This is the first, and most important, theorem in *projective geometry*: the study of relations between straight lines, with no distance or angular measurement.



Programs like *Geometer's Sketchpad*, or *GeoGebra*, or *C.A.R.* are excellent ways to attract young people to geometry, and hence to mathematics. They open up worlds of investigation and are powerful tools for design, research, and play.

Pascal's theorem

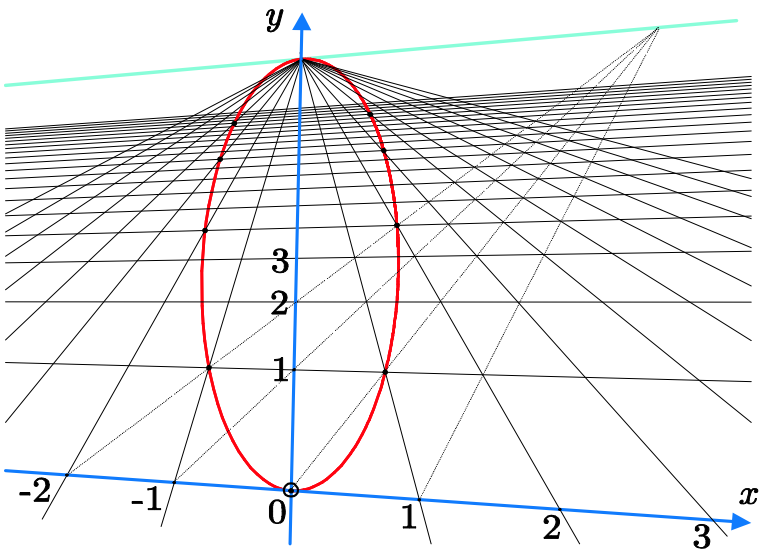
Pascal, as a teenager around 1640, discovered the beautiful fact that if $A_1, B_1, C_1, A_2, B_2, C_2$ lie on a conic (ellipse, parabola, hyperbola), then X, Y *and* Z *are still collinear*.



This is of considerable practical and theoretical importance: it allows for example one to create tools in Geometer's Sketchpad to draw a conic through any five general points.

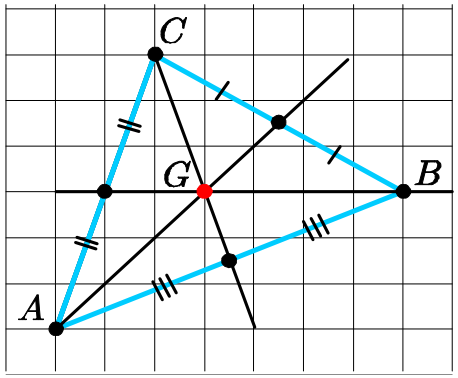
The parabola in perspective

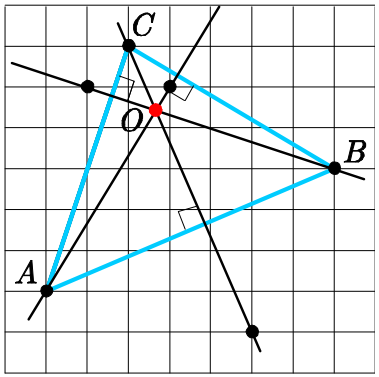
Here is a perspective view of the familiar parabola $y = x^2$. (You are standing on the usual x-y plane, looking out to where the horizon meets the y-axis). Note the points $[0,0]$, $[1,1]$, $[-1,1]$, $[2,4]$, $[-2,4]$, $[3,9]$, $[-3,9]$ on the parabola as usual.

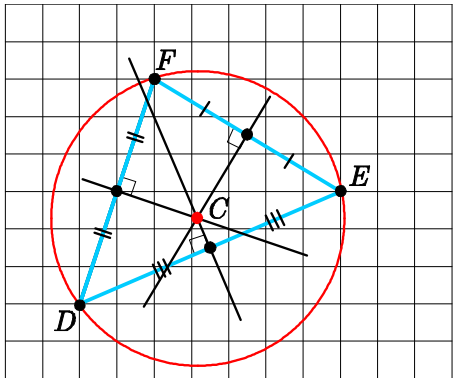


How to draw such a picture–accurately?? The 15th century Renaissance artists figured this out! The trick: *look at diagonals*.

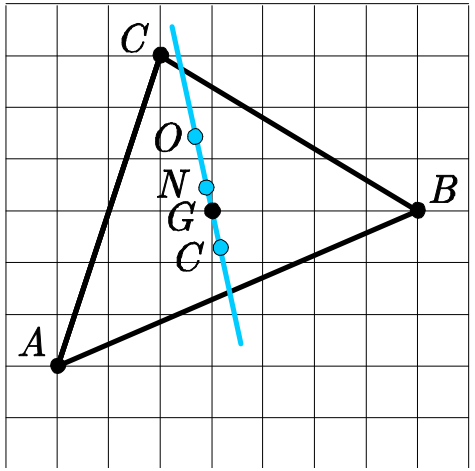
Centroid, Orthocenter, Circumcenter







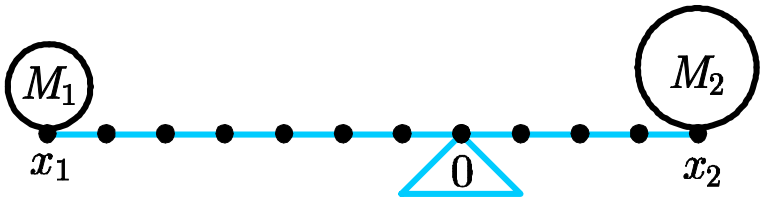
All three classical points are well-defined, and go back to the ancient Greeks. However there is a surprise—they are collinear, lying on the *Euler line*! This is well within range of average high school students: a bit of transformational geometry (dilations) is mostly what is needed. The fourth point is the *nine-point center* N, a good topic for gifted/enthusiastic students.



Barycentric Coordinates and the Principle of the Lever

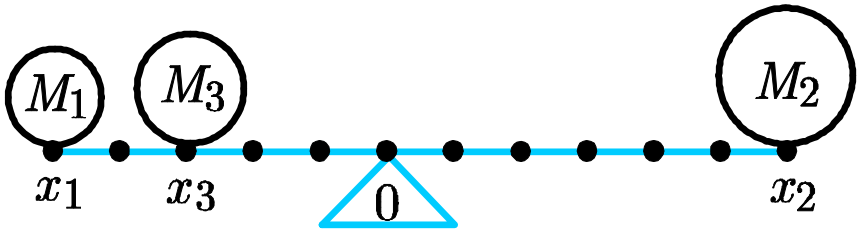
Archimedes law
asserts that if the masses balance then

$$M_1 x_1 + M_2 x_2 = 0.$$



This applies also to several masses, for example M_1, M_2 and M_3 at separations x_1, x_2 and x_3 balancing at O implies

$$M_1 x_1 + M_2 x_2 + M_3 x_3 = 0.$$

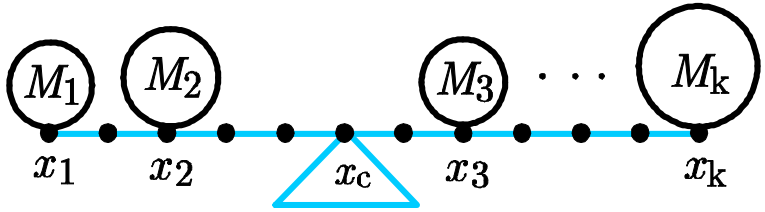


The relation would be

$$-5M_1 + 6M_2 - 3M_3 = 0.$$

Center of mass

A more general relation is that masses $M_1, M_2, \cdots, M_k$ at points $x_1, x_2, \cdots, x_k$ balance at the point x_c satisfying



$$Mx_c = M_1x_1 + M_2x_2 + \cdots + M_kx_k$$

where $M = M_1 + M_2 + \cdots + M_k$; the **center of mass** of the configuration. This extends to a continuous mass distribution:

$$Mx_c = \int_a^b m(x)\, x\, dx \qquad \text{where} \qquad M = \int_a^b m(x)\, dx$$

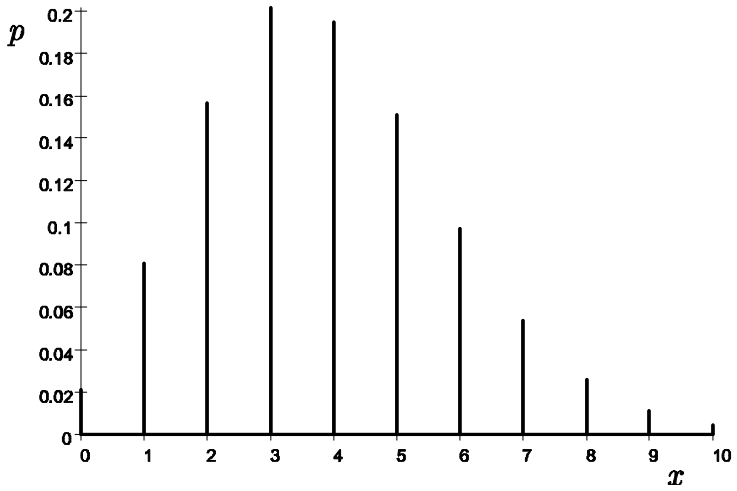
and also to the expected value/mean of a discrete or continuous probability distribution:

$$E(X) = \sum_{n=1}^{k} p_n x_n \qquad E(X) = \int_a^b p(x)\, x\, dx$$

where x_n occurs with probability p_k, or we have a probability distribution $p(x)$ on $[a, b]$.

Means of a Poisson distribution

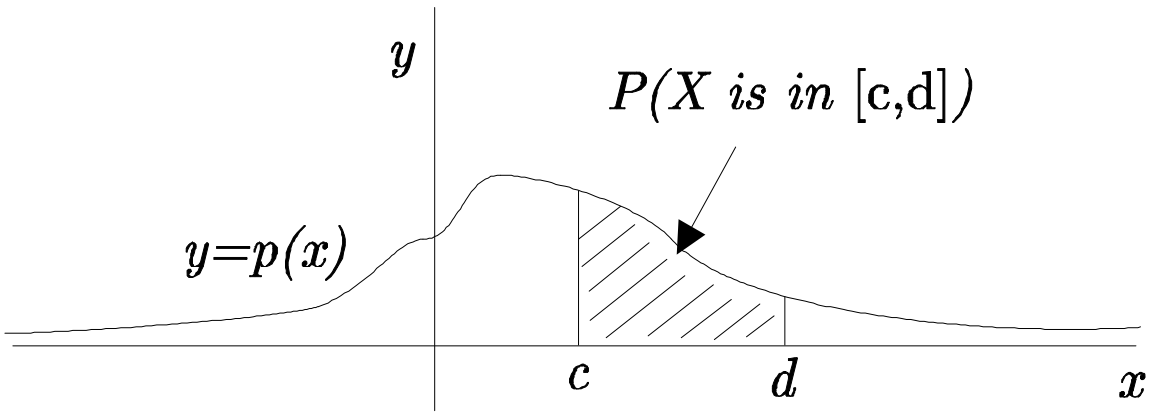
In a famous experiment in 1920, Rutherford and coworkers found the number of radioactive particles emmitted per standard time interval or a sample to be pretty close to a Poisson distribution. The mean turned out to be $\lambda = 3.870$, this is just the center of mass of this probability distribution, viewed as masses located above integer points.



In fact the Poisson distribution occurs quite frequently; count how many minutes you have to wait for the next bus at your favourite bus stop, and plot over many samples— this is something students can have fun doing!

Mean of continuous a distribution

Getting the hang of continuous distributions is much harder for students, where probabilities are associated to *areas* under the graph of a density function. In this case too the mean of a continuous random variable has a direct geometrical physical interpretation as the center of mass of an associated mass distribution.

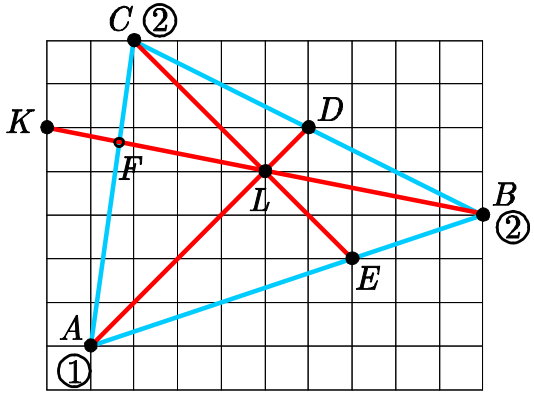


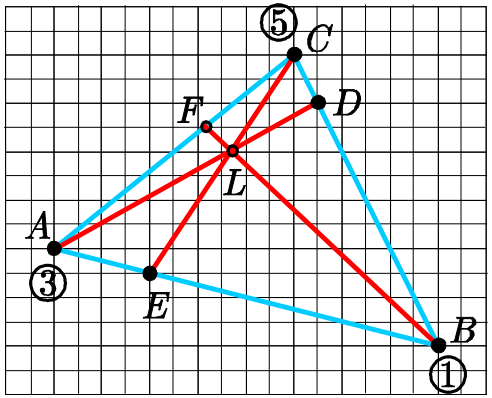
So geometrical and physical intuition is valuable even in statistics!

Making connections makes learning mathematics easier.

Center of mass in two dimensions

This extends to two dimensions: masses balance at a center of mass. In the first diagram $L = \frac{1}{5}A + \frac{2}{5}B + \frac{2}{5}C$.





This naturally connects with vector geometry! For the second diagram, the position of L is determined by the affine/convex combination

$$\overrightarrow{l} = \frac{3}{9}\overrightarrow{a} + \frac{1}{9}\overrightarrow{b} + \frac{5}{9}\overrightarrow{c}$$

independent of the position of the origin. These kind of convex/barycentric coordinates were introduced by Möbius (1820s), and are useful for describing regions in the plane, or space. They also connect to linear algebra.

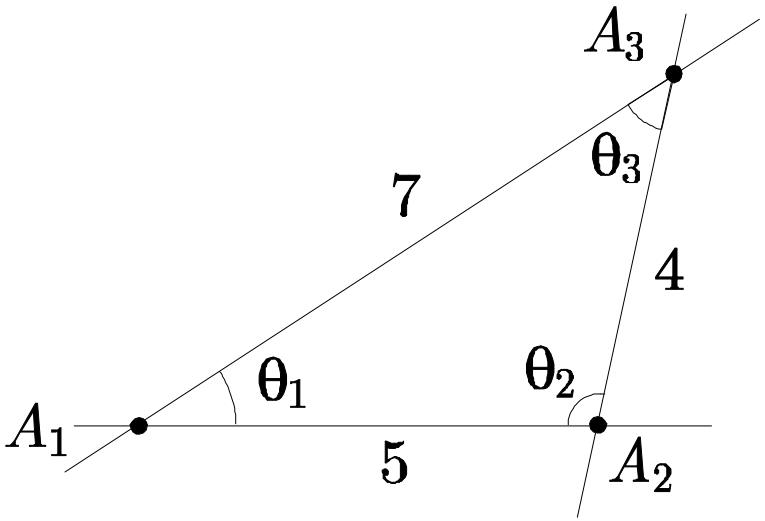
Olympic rankings and weighted totals

In 1908 the British press advocated a more balanced scoring for Olympics rather than 1) number of Gold or 2) total number of medals, with the following weights: *Gold=5*, *Silver=3* and *Bronze=1*. This gives a *weighted total* which is a more sensible indication of a countries performance, giving silver and bronze medalists also the glory that they deserve. Archimedes would have approved!

Rank	COUNTRY	G	S	B	Tot	W Tot	W Rank
1	United States	46	29	29	104	346	1
2	China	38	27	22	87	293	2
3	Great Britain	29	17	19	65	215	4
4	Russia	24	25	33	82	228	3
5	South Korea	13	8	7	28	96	7
6	Germany	11	19	14	44	126	5
7	France	11	11	12	34	100	6
8	Italy	8	9	11	28	78	10
9	Hungary	8	4	5	17	57	11
10	Australia	7	16	12	35	95	8*

Classical trigonometry

Classical trigonometry uses *lengths*, like $d_1 = 4$, $d_2 = 7$, $d_3 = 5$ and *angles*, like $\theta_1 \approx 33.92°$, $\theta_2 \approx 102.44°$ and $\theta_3 \approx 43.64°$.

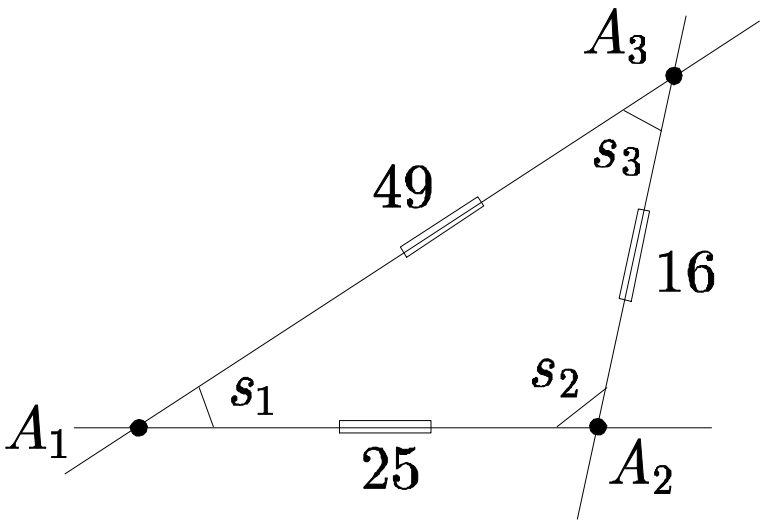


The laws are transcendental, like

$$\begin{aligned} d_3^2 &= d_1^2 + d_2^2 - 2d_1 d_2 \cos\theta_3 \\ \frac{\sin\theta_1}{d_1} &= \frac{\sin\theta_2}{d_2} = \frac{\sin\theta_3}{d_3} \\ \theta_1 + \theta_2 + \theta_3 &= 180°. \end{aligned}$$

Rational trigonometry

Rational trigonometry uses *quadrances*, like $Q_1 = 16$, $Q_2 = 49$, $Q_3 = 25$ and *spreads* like $s_1 = 384/1225$, $s_2 = 24/25$ and $s_3 = 24/49$.



The main laws become polynomial!:

$$\begin{aligned}
(Q_3 - Q_1 - Q_2)^2 &= 4Q_1Q_3(1 - s_3) \\
\frac{s_1}{Q_1} &= \frac{s_2}{Q_2} = \frac{s_3}{Q_3} \\
(s_1 + s_2 + s_3)^2 &= 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1s_2s_3.
\end{aligned}$$

[See my book *Divine Proportions: Rational Trigonometry to Universal Geometry* (2005) for details and applications].

Summary

Moral: The occasion of a national curriculum in maths is an opportunity to ask important questions about what we should be teaching, and how to attract more students to learn and be confident in their understanding, and even enjoyment!!, of mathematics. We want a balanced, interesting curriculum in line with Australia and world best practice. *Students* and *teachers* must come first, not political interest groups and powerful individuals with loud voices.

Geometry must be an important aspect of that discussion. There are lots of interesting, conceptual and useful topics in geometry that would appeal to a broad range of students, not necessarily at an elite level. We do need to keep an open mind about possible new directions and opportunities to strengthen our curriculum. But the ACARA Drafts on MM and SM... *are not it.*

Midterm Mark: (My personal scores only!)
ACARA Draft for *Mathematical Methods*: 3/10
ACARA Draft for *Specialist Mathematics*: 5/10.

More info

UNSW School of Mathematics Response to the ACARA Drafts on Mathematical Methods and Specialist Mathematics:
at *http://www.maths.unsw.edu.au/* under NEWS.

Wildbergers **YOUTUBE VIDEO SERIES** (at user: *njwildberger*):

- **WildTrig** (Rational Trigonometry)
- **MathHistory** (from ancient Greeks to 19th century)
- **MathFoundations** (setting up mathematics from the beginning!)
- **Elementary Maths K-6 Explained**

Thank you, and let's keep learning!