The 2014 Integration Bee—Solutions and comments

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Qualifying Round

1.
$$\int 3x^2(x^3-1)^4\,dx = \int u^4\ du = \frac{1}{5}u^5 + C = \frac{1}{5}(x^3-1)^5 + C.$$

2.
$$\int_{\sqrt{2}}^{\sqrt{5}} \frac{x}{\sqrt{x^2-1}}\ dx = \left[\sqrt{x^2-1}\right]_{\sqrt{2}}^{\sqrt{5}} = 2-1 = 1.$$

3.
$$\int \sin^2\left(\frac{x}{2}\right)\ dx = \int\ \frac{1}{2} - \frac{1}{2}\cos x\ dx = \frac{1}{2}x - \frac{1}{2}\sin x + C.$$

4.
$$\int \frac{1}{a^2+x^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

5.
$$\int \frac{3x-2}{2x+1}\ dx = \int \frac{3}{2} - \frac{7}{2}\frac{1}{2x+1}\ dx = \frac{3}{2}x - \frac{7}{4}\log|2x+1| + C.$$

6.
$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\theta}\ d\theta = \int_0^1 \frac{1}{1+\dfrac{2t}{1+t^2}}\cdot\frac{2}{1+t^2}\ dt$$
$$= \int_0^1 \frac{2}{(1+t)^2}\ dt = \left[-\frac{2}{1+t}\right]_0^1 = -1+2 = 1.$$

7.
$$\int x^{2014}\ dx = \frac{x^{2015}}{2015} + C.$$

8.
$$\int \frac{1}{x^2-4}\ dx = \int \frac{1}{4}\left(\frac{1}{x-2} - \frac{1}{x+2}\right)\ dx = \frac{1}{4}\log\left|\frac{x-2}{x+2}\right| + C.$$

Typeset by $\mathcal{AMS}$-TEX

9.

$$\int e^x(\cos x + \sin x)\ dx = Ae^x \cos x + Be^x \sin x + C$$

for some A, B. Differentiating with respect to x gives

$$e^x(\cos x + \sin x) = (A + B)e^x \cos x + (-A + B)e^x \sin x.$$

Comparing coefficients gives $A + B = 1, \quad -A + B = 1, \quad A = 0,\ B = 1$, so

$$\int e^x(\cos x + \sin x)\ dx = e^x \sin x + C.$$

Alternatively, you can do two integrations by parts.

10.

$$\int \frac{1}{x \log x}\ dx = \int \frac{1}{u}\ du = \log|u| + C = \log|\log x| + C.$$

11.

$$\int \pi^x\ dx = \int e^{x \log \pi}\ dx = \frac{1}{\log \pi} e^{x \log \pi} + C = \frac{1}{\log \pi} \pi^x + C.$$

12.

$$\int \frac{1}{3x^2}\ dx = \frac{1}{3} \cdot -\frac{1}{x} + C = -\frac{1}{3x} + C.$$

13.

$$\int_1^2 2x\sqrt{x^2 - 1}\ dx = \left[\frac{2}{3}(x^2 - 1)^{\frac{3}{2}}\right]_1^2 = \frac{2}{3}(3^{\frac{3}{2}} - 0) = 2\sqrt{3}.$$

14.

$$\int \frac{x^2 + 3}{x^2 - 3}\ dz = \frac{x^2 + 3}{x^2 - 3} z + C.$$

(We must assume x is a constant function of z.)

15.

$$\int x^5 \log|x|\ dx = \frac{x^6}{6} \log|x| - \int \frac{x^5}{6}\ dx = \frac{x^6}{6} \log|x| - \frac{x^6}{36} + C$$

by integration by parts with $u = \log|x|, \quad dv = x^5\,dx$.

16.

Let $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = I$. If we make the substitution $x = \pi - u$, we find

$$I = \int_0^{\pi} \frac{(\pi - u)\sin u}{1+\cos^2 u}\,du.$$

It follows that

$$2I = \int \frac{\pi \sin x}{1+\cos^2 x}\,dx = \pi \int_{-1}^{1} \frac{1}{1+u^2}\,du = \pi\Big[\tan^{-1} u\Big]_{-1}^{1} = \pi\left(\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right) = \frac{\pi^2}{2},$$

so

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = I = \frac{\pi^2}{4}.$$

17.

$$\int \sqrt{1+\sin 2x}\,dx = \int \sqrt{(\cos x + \sin x)^2}\,dx = \int \cos x + \sin x\,dx = -\cos x + \sin x + C.$$

(Not exactly legitimate, I'm afraid. It is not always the case that $\sqrt{v^2} = v$.) This is one reason why definite integrals make better quiz questions than indefinite integrals.

18.

$$\int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\cos^2 x}\,dx = \int_0^1 \frac{1}{2-u^2}\,du = \int_0^1 \frac{1}{2\sqrt{2}}\left(\frac{1}{\sqrt{2}-u} + \frac{1}{\sqrt{2}+u}\right)\,du$$
$$= \frac{1}{2\sqrt{2}}\left[\log\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right|\right]_0^1 = \frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{1}{\sqrt{2}}\log(1+\sqrt{2}).$$

19.

$$\int_{\frac{\pi}{7}}^{\frac{9\pi}{10}} \sqrt{\operatorname{cosec} x - \sin x}\,dx = \int_{\frac{\pi}{7}}^{\frac{9\pi}{10}} \sqrt{\frac{1}{\sin x} - \sin x}\,dx = \int_{\frac{\pi}{7}}^{\frac{9\pi}{10}} \sqrt{\frac{1-\sin^2 x}{\sin x}}\,dx$$
$$= \int_{\frac{\pi}{7}}^{\frac{9\pi}{10}} \frac{\cos x}{\sqrt{\sin x}}\,dx = \Big[2\sqrt{\sin x}\Big]_{\frac{\pi}{7}}^{\frac{9\pi}{10}} = 2\left(\sqrt{\sin\frac{9\pi}{10}} - \sqrt{\sin\frac{\pi}{7}}\right).$$

20.

$$\int \cot x \; dx = \int \frac{\cos x}{\sin x} \; dx = \log|\sin x| + C.$$

21.

$$\int_{-5}^{5} |\pi - x| \; dx = \int_{-5}^{\pi} \pi - x \; dx + \int_{\pi}^{5} x - \pi \; dx = \Big[\pi x - \frac{1}{2}x^2\Big]_{-5}^{\pi} + \Big[\frac{1}{2}x^2 - \pi x\Big]_{\pi}^{5}$$

$$= (\pi^2 - \frac{1}{2}\pi^2) - (-5\pi - \frac{25}{2}) + (\frac{25}{2} - 5\pi) - (\frac{1}{2}\pi^2 - \pi^2) = \pi^2 + 25.$$

Alternatively, from a graph,

$$\int_{-5}^{5} |pi - x| \; dx = \frac{1}{2}(\pi + 5)^2 + \frac{1}{2}(5 - \pi)^2 = \pi^2 + 25.$$

22.

$$\int e^{5 \log x} \; dx = \int x^5 \; dx = \frac{x^6}{6} + C.$$

23.

$$\int e^{e^x + x} \; dx = \int e^{e^x} \cdot e^x \; dx = \int e^u \; du = e^u + C = e^{e^x} + C.$$

24.

$$\int \sec x \tan x \; dx = \sec x + C.$$

(This is/used to be bookwork.)

25.

$$\int \frac{\log x}{x} \; dx = \int \frac{1}{u} \; du = \log|u| + C = \log|\log x| + C.$$

26.

$$\int \frac{1}{\sqrt{4 - x^2}} \; dx = \sin^{-1}\left(\frac{x}{2}\right) + C.$$

(This <u>certainly is</u> bookwork!

27.

$$\int \frac{1}{\cos x + 1} \; dx = \int \frac{1}{\frac{1 - t^2}{1 + t^2} + 1} \cdot \frac{2}{1 + t^2} \; dt = \int 1 \; dt = t + C = \tan^{-1}\left(\frac{x}{2}\right) + C.$$

28.

$$\int \operatorname{cosec}^2\theta \; d\theta = -\cot\theta + C.$$

This used to be bookwork once. Differentiate the right side to check!

29.

$$\int x^3 e^{3x^2} \; dx = \int \frac{1}{18}(3x^2)(6x)e^{3x^2} \; dx = \int \frac{1}{18} u e^u \; du = \frac{1}{18}\left(ue^u - e^u\right) + C$$
$$= \frac{1}{18}\left(3x^2 e^{3x^2} - e^{3x^2}\right) + C = \frac{1}{18} e^{3x^2}(3x^2 - 1) + C.$$

30.

$$\int \tan^{-1} x \; dx = \int u \sec^2 u \; du = u\tan u - \int \tan u \; du = u \tan u + \log|\cos u| + C$$
$$= u\tan u - \log|\sec u| + C = u\tan u - \frac{1}{2}\log(1 + \tan^2 u) + C$$
$$= x\tan^{-1} x - \frac{1}{2}\log(1 + x^2) + C.$$

Alternatively, do an integration by parts with $u = \tan^{-1} x, \; dv = dx$.

Group Stage

1.

$$\sin(A+B) = \sin A \cos B + \cos A \sin B,$$

so

$$\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \sin 5x \cos 3x \; dx = \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \frac{1}{2}(\sin 8x + \sin 2x) \; dx = \Big[\frac{1}{2}\left(-\frac{\cos 8x}{8} - \frac{\cos 2x}{2}\right)\Big]_{\frac{\pi}{8}}^{\frac{\pi}{4}}$$

$$= -\frac{1}{16}\Big[\cos 8x\Big]_{\frac{\pi}{8}}^{\frac{\pi}{4}} - \frac{1}{4}\Big[\cos 2x\Big]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = -\frac{1}{16}\left(1-(-1)\right) - \frac{1}{4}\left(0 - \frac{1}{\sqrt{2}}\right) = -\frac{1}{8} + \frac{1}{4\sqrt{2}}$$

$$= \frac{\sqrt{2}-1}{8}.$$

2.

$$\int_0^{16\pi} \sin^2 x \cos^2 x \; dx = \int_0^{16\pi} \frac{1}{4}\sin^2 2x \; dx = \int_0^{16\pi} \frac{1}{8}(1-\cos 4x) \; dx = \Big[\frac{1}{8}x - \frac{1}{32}\sin 4x\Big]_0^{16\pi}$$

$$= 2\pi.$$

3.

$$\int_{-5}^{5} |5-x| \; dx = \int_{-5}^{5} 5 - x \; dx = \Big[-\frac{1}{2}(5-x)^2\Big]_{-5}^{5} = \frac{1}{2}10^2 = 50.$$

Look at the graph!

4.

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{2-2x}{\sqrt{2x-x^2}} \; dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{2(1-x)}{\sqrt{1-(1-x)^2}} \; dx = \int_{\frac{1}{2}}^{-\frac{1}{2}} \frac{2u}{\sqrt{1-u^2}} \; du = 0,$$

because the integrand is odd and the interval of integration is balanced about the origin.

5.

$$\int_0^{\frac{1}{5}} \frac{2}{\sqrt{4-25x^2}} \; dx = \frac{2}{5}\int_0^{\frac{1}{5}} \frac{1}{\sqrt{\left(\frac{2}{5}\right)^2 - x^2}} \; dx = \frac{2}{5}\Big[\sin^{-1}\frac{5x}{2}\Big]_0^{\frac{1}{5}} = \frac{2}{5}\cdot\frac{\pi}{6} = \frac{\pi}{15}.$$

6.

$$\int \frac{\cos x}{4-\sin^2 x} \; dx = \int_0^1 \frac{1}{4-u^2} \; du = \int_0^1 \frac{1}{4}\left(\frac{1}{2+u} + \frac{1}{2-u}\right) \; du = \frac{1}{4}\Big[\log\Big|\frac{2+u}{2-u}\Big|\Big]_0^1$$

$$= \frac{1}{4}\log 3.$$

7.

$$\int_0^1 e^{\sin^2 x} e^{\cos^2 x}\ dx = \int_0^1 e\ dx = e.$$

8.

$$\int_0^1 \frac{1}{x^2 - 2x + 5}\ dx = \int_0^1 \frac{1}{(x-1)^2 + 2^2}\ dx = \left[\frac{1}{2}\tan^{-1}\frac{x-1}{2}\right]_0^1 = -\frac{1}{2}\tan^{-1}\left(-\frac{1}{2}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$

9.

$$\int_1^2 x^3 e^{x^2}\ dx = \int_1^2 \frac{1}{2}(2x)x^2 e^{x^2}\ dx = \int_1^4 \frac{1}{2}ue^u\ du = \left[\frac{1}{2}(ue^u - e^u)\right]_1^4 = \frac{3}{2}e^4.$$

10.

$$\int_0^{\frac{\pi}{4}} \sin x \sin 2x\ dx = \int_0^{\frac{\pi}{4}} \frac{1}{2}(\cos x - \cos 3x)\ dx = \left[\frac{\sin x}{2} - \frac{\sin 3x}{6}\right]_0^{\frac{\pi}{4}} = \frac{1}{2\sqrt{2}} - \frac{1}{6\sqrt{2}}$$
$$= \frac{2}{6\sqrt{2}} = \frac{1}{3\sqrt{2}}.$$

11.

$$\int_{\frac{1}{3}}^{\frac{1}{2}} \frac{e^{\frac{1}{x}}}{x^3}\ dx = \int_2^3 ue^u\ du = \left[ue^u - e^u\right]_2^3 = 2e^3 - e^2.$$

12.

$$\int_2^5 \cos(\sin x)\cos x\ dx = \int_{\sin 2}^{\sin 5} \cos u\ du = \left[\sin u\right]_{\sin 2}^{\sin 5} = \sin(\sin 5) - \sin(\sin 2).$$

13.

$$e^{\int (x^2 - 3x + 2)^{-1}\ dx} = e^{\int \frac{1}{x-1} - \frac{1}{x-2}\ dx} = e^{\log\left|\frac{x-1}{x-2}\right| + C} = e^C\left|\frac{x-1}{x-2}\right|.$$

14.

$$\int_0^1 e^x \tan e^x\ dx = \int_1^e \tan u\ du = \left[\log|\sec u|\right]_1^e = \log\left|\frac{\sec e}{\sec 1}\right|.$$

That's what a blind calculation gives, but it is not correct, would you believe! Sketch the graph of the integrand, and see what goes wrong! It makes you wonder about some of the other integrals!

15.

$$\int_0^1 xe^{-x}\ dx = \Big[-xe^{-x} - e^{-x}\Big]_0^1 = 1 - 2e^{-1}.$$

The indefinite integral can be guessed, or you can do an integration by parts.

16.

$$\int (e^x+1)^{20} e^x\ dx = \int_1^e (u+1)^{20}\ du = \left[\frac{1}{21}(u+1)^{21}\right]_1^2 = \frac{1}{21}\left((e+1)^{21} - 2^{21}\right).$$

17.

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x\ dx = \int_0^{\frac{1}{2}} u^2\ du = \left[\frac{1}{3}u^3\right]_0^{\frac{1}{2}} = \frac{1}{24}.$$

18.

$$\int_0^1 \frac{1}{1+x^2}\ dx = \Big[\tan^{-1} x\Big]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

19.

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \tan x + \cot x\ dx = \Big[\log|\sec x| + \log|\sin x|\Big]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = \Big[\log|\tan x|\Big]_{\frac{\pi}{12}}^{\frac{\pi}{4}} = -\log\tan\left(\frac{\pi}{12}\right)$$
$$= \log\cot\left(\frac{\pi}{12}\right).$$

20.

$$\int_1^2 \frac{1}{e^x+e^{-x}}\ dx = \int_1^2 \frac{e^x}{e^{2x}+1}\ dx = \int_e^{e^2} \frac{1}{u^2+1}\ du = \Big[\tan^{-1} u\Big]_e^{e^2}$$
$$= \tan^{-1} e^2 - \tan^{-1} e = \tan^{-1} \frac{e^2-e}{1+e^3}.$$

21

$$\int_a^b \frac{1}{x}\sec^2 \log x\ dx = \int_{\log a}^{\log b} \sec^2 u\ du = \Big[\tan u\Big]_{\log a}^{\log b} = \tan\log b - \tan\log a.$$

22.

$$\int_0^1 \sin^{-1} x + \cos^{-1} x \; dx = \int_0^1 \frac{\pi}{2} \; dx = \frac{\pi}{2}.$$

Draw the graph!

23.

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \sec x \; dx = \Big[\log|\sec x + \tan x| \Big]_{\frac{\pi}{3}}^{\frac{\pi}{4}} = \log(\sqrt{2}+1) - \log(2+\sqrt{3}) = \log\left(\frac{\sqrt{2}+1}{\sqrt{3}+2}\right)$$

$$= -\log\left(\frac{\sqrt{3}+2}{\sqrt{2}+1}\right),$$

which is negative, because of the unusual limits on the integral.

24.

$$\int_1^4 \frac{x^3 - x}{x^{\frac{3}{2}}} \; dx = \int_1^4 x^{\frac{3}{2}} - x^{-\frac{1}{2}} \; dx = \Big[\frac{2}{5} x^{\frac{5}{2}} - 2x^{\frac{1}{2}}\Big]_1^4 = \frac{2}{5}(32-1) - 2(2-1) = \frac{52}{5}.$$

25.

$$\int_{\frac{\pi^2}{9}}^{\frac{\pi^2}{4}} \frac{\sin\sqrt{x}}{\sqrt{x}} \; dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} 2\sin u \; du = \Big[-2\cos u \Big]_{\frac{\pi}{3}}^{\frac{\pi}{2}} = -2\left(0 - \frac{1}{2}\right) = 1.$$

26.

$$\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x} \; dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x} \; dx = 1.$$

(Done earlier!)

27.

$$\int_1^e \frac{x+49}{x-49} \; dx = \int_1^e 1 + \frac{98}{x-49} \; dx = \Big[x + 98\log|x-49| \Big]_1^e = e - 1 + 98\log\left|\frac{e-49}{1-49}\right|$$

$$= e - 1 + 98\log\left(\frac{49-e}{48}\right).$$

Draw the graph!

28.

$$\int_5^7 2^{\log x}\ dx = \int_5^7 x^{\log 2}\ dx = \left[\frac{x^{\log 2+1}}{\log 2+1}\right]_5^7 = \frac{1}{\log 2+1}\left(7^{\log 2+1} - 5^{\log 2+1}\right).$$

29.

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sec x + \tan x \sin x}\ dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin^2 x}\ dx = \int_0^1 \frac{1}{1+u^2}\ du = \left[\tan^{-1} u\right]_0^1 = \frac{\pi}{4}.$$

30.

$$\int_1^e \frac{1-\log x}{x^2}\ dx = \int_0^1 \frac{1-t}{e^{2t}} e^t\ dt = \int_0^1 (1-t)e^{-t}\ dt = \left[te^{-t}\right]_0^1 = e^{-1}.$$

31.

$$\int_0^{\frac{\pi}{3}} \sec^4 x \tan x\ dx = \int_1^2 u^3\ du = \frac{1}{24}.$$

32.

$$\int_0^1 x^2(x^3+1)^4\ dx = \int_1^2 \frac{1}{3}u^4\ du = \left[\frac{u^5}{15}\right]_1^2 = \frac{31}{15}.$$

33.

$$\int_0^{\frac{2}{\pi}} \sin^2 x\ dx = \int_0^{\frac{2}{\pi}} \frac{1}{2} - \frac{1}{2}\cos 2x\ dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{2}{\pi}} = \frac{1}{\pi} - \frac{1}{4}\sin\left(\frac{4}{\pi}\right).$$

34.

$$\int_e^\pi \frac{\cos 3x}{4\cos^3 x - 3\cos x}\ dx = \int_e^\pi 1\ dx = \pi - e.$$

(You have to know that $\cos 3x = 4\cos^3 x - 3\cos x$. Try to prove it.)

35.

$$\frac{2}{\pi}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \tan^{-1}\theta + \cot^{-1}\theta\ d\theta = \frac{2}{\pi}\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} 1\ d\theta = \frac{2}{\pi}\cdot\frac{\pi}{8} = \frac{1}{4}.$$

Unusual. $\tan^{-1}$ and $\cot^{-1}$ are usually applied to x, not θ.

36.

$$\int_0^{\frac{\pi}{4}} \frac{e^{\tan x}}{\cos^2 x}\ dx = \int_0^1 e^u\ du = e - 1.$$

37.

$$\int_{\sqrt{\frac{\pi}{6}}}^{\sqrt{\frac{\pi}{2}}} x\cos x^2\ dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\cos u\ du = \left[\frac{1}{2}\sin u\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}} = \frac{1}{2}\left(1 - \frac{1}{2}\right) = \frac{1}{4}.$$

38.

$$\int_1^{\sec^2\left(\frac{\pi}{4}\right)-1} e^x\ dx = \int_1^1 e^x\ dx = 0.$$

39.

$$\int_0^{\log 3} \frac{e^x}{1+e^x}\ dx = \int_2^4 \frac{1}{u}\ du = \log 4 - \log 2 = \log 2.$$

40.

$$\int_{-1}^0 \frac{2x+1}{x^2+2x+2}\ dx = \int_0^1 \frac{2u-1}{u^2+1}\ du = \Big[\log(u^2+1) - \tan^{-1} u\Big]_0^1 = \log 2 - \frac{\pi}{4}.$$

41.

$$\int_4^{12} \frac{1}{(4+x)\sqrt{x}}\ dx = \int_2^{\sqrt{12}} \frac{1}{(4+u^2)u} 2u\ du = \int_2^{\sqrt{12}} \frac{2}{4+u^2}\ du = \left[\tan^{-1}\left(\frac{u}{2}\right)\right]_2^{\sqrt{12}}$$
$$= \tan^{-1}\sqrt{3} - \tan^{-1} 1 = \frac{\pi}{3} - \frac{\pi}{4} = \frac{\pi}{12}.$$

42.

$$\int_1^3 \frac{2x^2 - x + 1}{x-2}\ dx$$

does not exist, because of the discontinuity at $x = 2$.

43.

$$\int_{-\sqrt{3}}^{\sqrt{3}} \frac{-1}{\sqrt{4-x^2}}\ dx = \left[-\sin^{-1}\left(\frac{x}{2}\right)\right]_{-\sqrt{3}}^{\sqrt{3}} = -\left(\frac{\pi}{3} - \left(-\frac{\pi}{3}\right)\right) = -\frac{2\pi}{3}.$$

44.

$$\int_0^{\frac{\pi}{4}} \sin^2 x \, dx = \Big[\frac{1}{2}x - \frac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{4}} = \frac{\pi}{8} - \frac{1}{4}.$$

45.

$$\int_0^1 \frac{\sqrt{x}}{1+x} \, dx = \int_0^1 \frac{u}{1+u^2} 2u \, du = \int_0^1 \frac{2u^2}{u^2+1} \, du = \int_0^1 2 - \frac{2}{u^2+1} \, du$$
$$= \Big[2u - 2\tan^{-1} u\Big]_0^1 = 2 - \frac{\pi}{2}.$$

46.

$$\int_1^2 (x+1)^{-2} \, dx = \Big[-\frac{1}{x+1}\Big]_1^2 = -\frac{1}{3} + \frac{1}{2} = \frac{1}{6}.$$

47.

$$\int_2^4 (\sqrt{x}-1)^2 \, dx = \int_{\sqrt{2}}^2 (u-1)^2 2u \, du = \int_{\sqrt{2}}^2 2u^3 - 4u^2 + 2u \, du = \Big[\frac{1}{2}u^4 - \frac{4}{3}u^3 + u^2\Big]_{\sqrt{2}}^2$$
$$= \frac{1}{2}(16-4) - \frac{4}{3}(8 - 2\sqrt{2}) + (4-2) = 6 - \frac{4}{3}(8 - 2\sqrt{2} + 2 = -\frac{8}{3} + \frac{8}{3}\sqrt{2} = \frac{8}{3}(\sqrt{2}-1).$$

48.

$$\int_2^{-1} x^3 - 2x^2 + 3x - 4 \, dx = \Big[\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{3}{2}x^2 - 4x\Big]_2^{-1}$$
$$= \frac{1}{4}\left((-1)^4 - 2^4\right) - \frac{2}{3}\left((-1)^3 - 2^3\right) + \frac{3}{2}\left((-1)^2 - 2^2\right) - 4\left(-1-2\right)$$
$$= \frac{1}{4} \times -15 - \frac{2}{3} \times -9 + \frac{3}{2} \times -3 - 4 \times -3 = -\frac{15}{4} + 6 - \frac{9}{2} + 12 = 18 - \frac{15}{4} - \frac{18}{4}$$
$$= 18 - \frac{33}{4} = \frac{39}{4}.$$

Quarter–Finals

1.

$$\int \frac{1}{\sqrt{\sin^3 x \cos x}}\,dx = \int \frac{1}{\sqrt{\tan^3 x/\sec^4 x}}\,dx = \int \frac{\sec^2 x}{(\tan x)^{\frac{3}{2}}}\,dx = -2(\tan x)^{-\frac{1}{2}} + C$$
$$= -\frac{2}{\sqrt{\tan x}} + C.$$

2.

$$\int \frac{\sqrt{\tan x}}{\sin x \cos x}\,dx = \int \frac{\sqrt{\tan x}}{\tan x/\sec^2 x}\,dx = \int \frac{\sec^x}{(\tan x)^{\frac{1}{2}}}\,dx = 2(\tan x)^{\frac{1}{2}} + C$$
$$= 2\sqrt{\tan x} + C.$$

3.

$$\int \frac{\sin^5 x}{\cos x}\,dx = \int \frac{\sin x(1-\cos^2 x)^2}{\cos x}\,dx = \int \frac{\sin x(1 - 2\cos^2 x + \cos^4 x)}{\cos x}\,dx$$
$$= \int \tan x - 2\cos x \sin x + \cos^3 x \sin x\,\,dx = \log|\sec x| + \cos^2 x - \frac{1}{4}\cos^4 x + C.$$

4.

$$\int \cos\sqrt{x}\,dx = \int 2u\cos u\,du = 2u\sin u + 2\cos u + C = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

5.

$$\int \frac{3x-1}{(x^2+1)(x^2+2)}\,dx = \int (3x-1)\left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)dx = \int \frac{3x-1}{x^2+1} - \frac{3x-1}{x^2+2}\,dx$$
$$= \frac{3}{2}\log(x^2+1) - \tan^{-1} x - \frac{3}{2}\log(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C.$$

6.

$$\int \frac{x}{x^4+1}\,dx = \int \frac{1}{2}\frac{1}{u^2+1}\,du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1} x^2 + C.$$

7.

$$\int \frac{\log \cos x}{\cot x}\ dx = \int -\frac{\log u}{u}\ du = \int -v\ dv = -\frac{1}{2}v^2 + C = -\frac{1}{2}(\log|u|)^2 + C$$
$$= -\frac{1}{2}(\log|\cos x|)^2 + C.$$

8.

$$\int \sin^5 x \cos^6 x\ dx = \int \cos^6 x(1-\cos^2 x)^2 \sin x\ dx = -\frac{1}{7}\cos^7 x+\frac{2}{9}\cos^9 x-\frac{1}{11}\cos^{11} x+C.$$

9.

$$\int \frac{1}{\sqrt{x^2+2x}}\ dx = \int \frac{1}{\sqrt{(x+1)^2-1}}\ dx = \int \frac{1}{\sqrt{u^2-1}}\ du$$
$$= \int \frac{1}{\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\ d\theta = \int \sec\theta\ d\theta = \log|\sec\theta+\tan\theta| + C$$
$$= \log|u+\sqrt{u^2-1}| + C = \log|x+1+\sqrt{x^2+2x}| + C.$$

10.

$$\int (3+2x)\log x\ dx = \int (3+2e^t)te^t\ dt = \int 3te^t + 2te^{2t}\ dt$$
$$= 3te^t - 3e^t + te^{2t} - \frac{1}{2}e^{2t} + C = 3x\log|x| - 3x + x^2\log|x| - \frac{1}{2}x^2 + C$$
$$= (x^2+3x)\log|x| - \frac{1}{2}x^2 - 3x + C.$$

11.

$$\int \frac{12}{(x^2+4)(x^2+16)}\ dx = \int \frac{1}{x^2+4} - \frac{1}{x^2+16}\ dx = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{4}\tan^{-1}\left(\frac{x}{4}\right) + C.$$

12.

If for the moment we pretend $\sqrt{x^2} = x$, we find

$$\int \sqrt{x^2+x^4}\ dx = \int x\sqrt{1+x^2}\ dx = \int \frac{1}{2}u^{\frac{1}{2}}\ du = \frac{1}{3}u^{\frac{3}{2}} + C = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C$$
$$= \frac{1}{3x^3}(x^2+x^4)^{\frac{3}{2}} + C.$$

Luckily, since this function is odd (apart from the C), it is correct.

13.

$$\int \sqrt{1-x^2}\ dx = \int \sqrt{1-\sin^2\theta}\cos\theta\ d\theta = \int \cos^2\theta\ d\theta = \int \frac{1}{2}+\frac{1}{2}\cos 2\theta\ d\theta$$
$$= \frac{1}{2}+\frac{1}{4}\sin 2\theta + C = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C = \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2}+C.$$

Actually, a similar comment applies to this integral as to the previous one!

14.

$$\int \frac{1}{(4+x^2)^{\frac{3}{2}}}\ dx = \int \frac{1}{(4+4\tan^2\theta)^{\frac{3}{2}}}\cdot 2\sec^2\theta\ d\theta = \int \frac{1}{8\sec^3\theta}\cdot 2\sec^2\theta\ d\theta$$
$$= \int \frac{1}{4}\cos\theta\ d\theta = \frac{1}{4}\sin\theta + C = \frac{1}{4}\frac{x}{\sqrt{4+x^2}}+C.$$

15.

$$\int \frac{1}{\sqrt{e^{2x}-1}}\ dx = \int \frac{e^x}{e^x\sqrt{e^{2x}-1}}\ dx = \int \frac{1}{u\sqrt{u^2-1}}\ du$$
$$= \int \frac{1}{\sec\theta\tan\theta}\sec\theta\tan\theta\ d\theta = \int 1\ d\theta = \theta + C = \sec^{-1}u + C$$
$$= \sec^{-1}e^x + C.$$

Alternatively, $= \tan^{-1}\sqrt{e^{2x}-1}+C$.

16.

$$\int \cos^4 x\ dx = \int \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right)^2\ dx = \int \frac{1}{4}\left(1+2\cos 2x+\cos^2 2x\right)\ dx$$
$$= \int \frac{1}{4}\left(1+2\cos 2x+\frac{1}{2}+\frac{1}{2}\cos 4x\right)\ dx = \int \frac{3}{8}+\frac{1}{2}\cos 2x+\frac{1}{8}\cos 4x\ dx$$
$$= \frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x + C = \frac{3}{8}x+\frac{1}{2}\sin x\cos x+\frac{1}{16}\sin 2x\cos 2x + C$$
$$= \frac{3}{8}x+\frac{1}{2}\sin x\cos x+\frac{1}{8}\sin x\cos x(2\cos^2 x-1)+C$$
$$= \frac{3}{8}x+\frac{3}{8}\sin x\cos x+\frac{1}{4}\sin x\cos^3 x+C.$$

17.

$$\int \frac{1}{\sqrt{9+16x-4x^2}}\ dx = \int \frac{1}{2\sqrt{\frac{9}{4}+4x-x^2}}\ dx = \int \frac{1}{2\sqrt{\frac{25}{4}-(x-2)^2}}\ dx$$

$$= \int \frac{1}{2\sqrt{\left(\frac{5}{2}\right)^2-(x-2)^2}}\ dx = \frac{1}{2}\sin^{-1}\left(\frac{2}{5}(x-2)\right)+C.$$

18.

$$\int \sin\log x\ dx = \int e^u \sin u\ du = e^u(\sin u - \cos u) + C = x(\sin\log x - \cos\log x) + C.$$

(See question 9 of the Qualifying Round.)

19.

$$\int 2x\tan^{-1}x\ dx = x^2\tan^{-1}x - \int \frac{x^2}{1+x^2}\ dx = x^2\tan^{-1}x - \int 1 - \frac{1}{1+x^2}\ dx$$

$$= x^2\tan^{-1}x - x + \tan^{-1}x + C = (1+x^2)\tan^{-1}x - x + C.$$

20.

$$\int \frac{x^3e^{x^2}}{(x^2+1)^2}\ dx = \int \frac{1}{2}\frac{ue^u}{(u+1)^2}\ du = \frac{1}{2}\int \frac{(v-1)e^{v-1}}{v^2}\ dv = \frac{1}{2e}\int \frac{(v-1)e^v}{v^2}\ dv$$

$$= \frac{1}{2e}\left\{(v-1)e^v\cdot -\frac{1}{v} - \int -\frac{1}{v}\cdot ve^v\ dv\right\} = \frac{1}{2e}\left\{-e^v + \frac{e^v}{v} + \int e^v\ dv\right\} = \frac{1}{2e}\frac{e^v}{v} + C$$

$$= \frac{1}{2e}\frac{e^{u+1}}{u+1} + C = \frac{1}{2}\frac{e^u}{u+1} + C = \frac{1}{2}\frac{e^{x^2}}{x^2+1} + C = \frac{e^{x^2}}{2(x^2+1)} + C.$$

21.

$$\int (x-1)(x+1)^{11}\ dx = \int u^{11}(u-2)\ du = \int u^{12} - 2u^{11}\ du = \frac{1}{13}u^{13} - \frac{1}{6}u^{12} + C$$

$$= \frac{1}{13}(x+1)^{13} - \frac{1}{6}(x+1)^{12} + C = \frac{1}{78}(x+1)^{12}\left(6(x+1)-13\right) + C$$

$$= \frac{1}{78}(x+1)^{12}(6x-7) + C..$$

22.

$$\int \frac{1}{1+x^{\frac{1}{4}}}\ dx = \int \frac{1}{1+u}4u^3\ du = \int \frac{4}{v}(v-1)^3\ dv = \int 4(v^2-3v+3-\frac{1}{v})\ dv$$

$$= 4(\frac{1}{3}v^3 - \frac{3}{2}v^2 + 3v - \log|v|) + C = \frac{4}{3}(u+1)^3 - 6(u+1)^2 + 12(u+1) - 4\log|u+1| + C$$
$$= \frac{4}{3}u^3 - 2u^2 + 4u - 4\log|u+1| + C = \frac{4}{3}x^{\frac{3}{4}} - 2x^{\frac{1}{2}} + 4x^{\frac{1}{4}} - 4\log|1+x^{\frac{1}{4}}| + C.$$

23.

$$\int \frac{\sin^5(x^{-1})\cos(x^{-1})}{x^2}\ dx = \int -\sin^5 u \cos u\, du = -\frac{1}{6}\sin^6 u + C = -\frac{1}{6}\sin^6(x^{-1}) + C$$

24.

$$\int \frac{2}{\left(\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\right)^2}\ dx = \int \sec^2\left(x+\frac{\pi}{4}\right) dx = \tan\left(x+\frac{\pi}{4}\right) + C.$$

25.

$$\int \left(\sin^{-1} x\right)^2\ dx = \int u^2 \cos u\ du = u^2 \sin u + 2u\cos u - 2\sin u + C$$
$$= x\left(\sin^{-1} x\right)^2 + 2\sqrt{1-x^2}\sin^{-1} x - 2x + C.$$

Semi–Finals

1.

$$\int \frac{\sec^2 x}{\sec x + \tan x}\ dx = \int \sec^2 x\,(\sec x - \tan x)\ \ dx = \int \sec^3 x\ dx - \frac{1}{2}\tan^2 x + C.$$

The notoriously ugly $\int \sec^3 x\ dx$ can be calculated by integration by parts.

$$\int \sec^3\theta\ d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| + C.$$

So our integral

$$= \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\log|\sec\theta + \tan\theta| - \frac{1}{2}\tan^2\theta + C$$
$$= \frac{1}{2}\tan\theta(sec\theta - \tan\theta) + \frac{1}{2}\log|\sec\theta + \tan\theta| + C.$$

2.

$$\int e^{x^x}(\log x+1)x^{2x}\ dx = \int ue^u\ du = ue^u - e^u + C = x^x e^{x^x} - e^{x^x} + C = e^{x^x}(x^x - 1) + C.$$

3.

$$\int \frac{2x - 9\sqrt{x} + 9}{(x - 3\sqrt{x})^{\frac{1}{3}}}\ dx = \int \frac{2u^2 - 9u + 9}{(u^2 - 3u)^{\frac{1}{3}}}2u\ du = \int \frac{4u^3 - 18u^2 + 18u}{(u^2 - 3u)^{\frac{1}{3}}}\ du$$
$$= \int \frac{(u^2 - 3u)(4u - 6)}{(u^2 - 3u)^{\frac{1}{3}}}\ du = \int (u^2 - 3u)^{\frac{2}{3}}2(2u - 3)\ du = \int 2v^{\frac{2}{3}}\ dv$$
$$= \frac{6}{5}v^{\frac{5}{3}} + C = \frac{6}{5}(u^2 - 3u)^{\frac{5}{3}} + C = \frac{6}{5}(x - 3\sqrt{x})^{\frac{5}{3}} + C.$$

4.

$$\int \frac{x^5}{\sqrt{1 + x^2}}\ dx = \int \frac{1}{2}\frac{(u - 1)^2}{\sqrt{u}}\ du = \int \frac{1}{2}\left(u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right)\ du$$
$$= \frac{1}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + u^{\frac{1}{2}} + C = \frac{1}{15}u^{\frac{1}{2}}(3u^2 - 10u + 15) + C$$
$$= \frac{1}{15}(3(x^2 + 1)^2 - 10(x^2 + 1) + 15)\sqrt{1 + x^2} + C = \frac{1}{15}(3x^4 - 4x^2 + 8)\sqrt{1 + x^2} + C.$$

5.

$$\int \frac{\log(1+\log x)}{x}\ dx = \int \log u\ du = u\log u - u + C$$
$$= (1+\log x)\log(1+\log x) - \log x + C.$$

6.

$$\int \frac{x^{\frac{1}{2}}}{x^{\frac{1}{3}}+x^{\frac{1}{4}}}\ dx = \int \frac{u^6}{u^4+u^3}12u^{11}\ du = \int \frac{12u^{14}}{u+1}\ du = \int \frac{12(u^{14}-1)+12}{u+1}\ du$$
$$= \int 12\Big(u^{13}-u^{12}+u^{11}-u^{10}+u^9-u^8+u^7-u^6+u^5-u^4$$
$$+u^3-u^2+u-1+\frac{1}{u+1}\Big)\ du$$
$$= \frac{6}{7}u^{14} - \frac{12}{13}u^{13} + u^{12} - \frac{12}{11}u^{11} + \frac{6}{5}u^{10} - \frac{4}{3}u^9 + \frac{3}{2}u^8 - \frac{12}{7}u^7 + 2u^6 - \frac{12}{5}u^5$$
$$+3u^4 - 4u^3 + 6u^2 - 12u + 12\log|u+1| + C$$

Now set $u = x^{\frac{1}{12}}$.

7.

$$\int \frac{x^3+3x^2+3x}{x^4+4x^3+6x^2+4x+1}\ dx = \int \frac{(x+1)^3-1}{(x+1)^4}\ dx = \int \frac{1}{x+1} - \frac{1}{(x+1)^4}\ dx$$
$$= \log|x+1| + \frac{1}{3(x+1)^3} + C.$$

8.

$$\int \sec x \operatorname{cosec} x\ dx = \int \frac{1}{\cos x\ \sin x}\ dx = \int \frac{2}{\sin 2x}\ dx = \int 2\operatorname{cosec} 2x\ dx$$
$$= -\log|\operatorname{cosec} 2x + \cot 2x| + C.$$

9.

$$\int \frac{1}{x+x^4}\ dx = \int \frac{1}{x(x+1)(x^2-x+1)}\ dx = \int \frac{A}{x} + \frac{B}{x+1} + \frac{Cx+D}{x^2-x+1}\ dx$$
$$= A\log|x| + B\log|x+1| + \frac{C}{2}\log(x^2-x+1) + \left(D+\frac{C}{2}\right)\sin^{-1}\left(\frac{2}{\sqrt{3}}\left(x-\frac{1}{2}\right)\right) + K,$$

where $A,\ B,\ C$ and D are given by

$$A(x+1)(x^2-x+1)+Bx(x^2-x+1)+(Cx+D)x(x+1)=1$$

(for all x).

We have

$$A+B+C=0,\quad -B+C+D=0,\quad B+D=0,\quad A=1,$$

so

$$A=1,\quad B=-\frac{1}{3},\quad C=-\frac{2}{3},\quad D=\frac{1}{3},$$

and the integral is

$$=\log|x|-\frac{1}{3}\log|x+1|-\frac{1}{3}\log(x^2-x+1)+K=\log|x|-\frac{1}{3}\log|x^3+1|+K$$
$$=\frac{1}{3}\log\left|\frac{x^3}{x^3+1}\right|+K=\frac{1}{3}\log\left|\frac{x^4}{x+x^4}\right|+K.$$

10.

$$\int \operatorname{cosec} x\ dx=-\log|\operatorname{cosec} x+\cot x|+C.$$

(Pure bookwork.)

11.

$$\int \cos\sin\sin x\cos\sin x\cos x\ dx=\sin\sin\sin x+C.$$

(Obvious!)

12.

$$\int\frac{x}{\sqrt{1-2x-x^2}}\ dx=\int\frac{x}{\sqrt{2-(x+1)^2}}\ dx=\int\frac{u-1}{\sqrt{2-u^2}}\ du$$
$$=-\sqrt{2-u^2}-\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)+C=-\sqrt{1-2x-x^2}-\sin^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+C.$$

13.

$$\int x^3\log(5x)\ dx=\int\frac{1}{5^4}u^3\log u\ du=\int\frac{1}{5^4}ve^{4v}\ dv=\frac{1}{5^4}\left(\frac{1}{4}ve^{4v}-\frac{1}{16}e^{4v}\right)+C$$
$$=\frac{1}{5^4}\left(\frac{1}{4}u^4\log u-\frac{1}{16}u^4\right)+C=\frac{1}{4}x^4\log 5x-\frac{1}{16}x^4+C.$$

14.

$$\int \left(\frac{\log x}{x}\right)^2 \, dx = \int \left(\frac{u}{e^u}\right)^2 e^u \, du = \int u^2 e^{-u} \, du$$

$$= -u^2 e^{-u} - 2ue^{-u} - 2e^{-u} + C = -\frac{(\log x)^2}{x} - 2\frac{\log x}{x} - \frac{2}{x} + C.$$

15. If n is an integer, $n \geq 2$,

$$\int \frac{1}{x^n + x} \, dx = \int \frac{1}{x(x^{n-1}+1)} \, dx = \int \frac{x^{n-2}}{x^{n-1}(x^{n-1}+1)} \, dx$$

$$= \int x^{n-2} \left(\frac{1}{x^{n-1}} - \frac{1}{x^{n-1}+1}\right) \, dx$$

$$= \int \frac{1}{x} - \frac{x^{n-2}}{x^{n-1}+1} \, dx = \log|x| - \frac{1}{n-1} \log|x^{n-1}+1| + C.$$

Finals

1.

$$\int \sin^2 x \cos^4 x \; dx = \int \left(\frac{1}{2}\sin 2x\right)^2 \left(\frac{1}{2}+\frac{1}{2}\cos 2x\right) \; dx = \frac{1}{8}\int \sin^2 2x + \sin^2 2x \cos 2x \; dx$$

$$= \frac{1}{8}\int \frac{1}{2} - \frac{1}{2}\cos 4x + \sin^2 2x \cos 2x \; dx = \frac{1}{16}x - \frac{1}{64}\sin 4x + \frac{1}{48}\sin^3 2x + C.$$

This solution was indicated to me by one of the competitors.

2.

$$\int \frac{4x^2-15x+29}{(x-5)(x^2-4x+13)} \; dx = \int \frac{A}{x-5} + \frac{Bx+C}{(x-2)^2+3^2} \; dx$$

$$= A\log|x-5| + \frac{B}{2}\log(x^2-4x+13) + \frac{2B+C}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + K,$$

where A, B and C are given by

$$A(x^2-4x+13) + (Bx+C)(x-5) = 4x^2-15x+29$$

(for all x).

We have

$$A+B=4, \quad -4A-5B+C=-15, \quad 13A-5C=29,$$

so

$$A=3, \quad B=1, \quad C=2$$

and the integral is

$$= 3\log|x-5| + \frac{1}{2}\log(x^2-4x+13) + \frac{4}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + K.$$

3.

$$\int \sqrt{1+\sin x}\cot x \; dx = \int \frac{\sqrt{1+\sin x}}{\sin x}\cos x \; dx = \int \frac{\sqrt{1+u}}{u} \; du$$

$$= \int \frac{v}{v^2-1}2v \; dv = \int \frac{2v^2}{v^2-1} \; dv = \int 2 + \frac{2}{v^2-1} \; dv = \int 2 + \left(\frac{1}{v-1} - \frac{1}{v+1}\right) \; dv$$

$$= 2v + \log\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+u} + \log\left|\frac{\sqrt{1+u}-1}{\sqrt{1+u}+1}\right| + C$$

$$= 2\sqrt{1+\sin x} + \log\left|\frac{\sqrt{1+\sin x}-1}{\sqrt{1+\sin x}+1}\right| + C.$$

4.

$$\int \frac{\sin 2x}{1+2\sin^2 x}\ dx = \int \frac{\sin 2x}{2-\cos 2x}\ dx = \int -\frac{1}{2}\frac{1}{2-u}\ du = \frac{1}{2}\log|2-u| + C$$
$$= \frac{1}{2}\log|2-\cos 2x| + C = \frac{1}{2}\log(2-\cos 2x) + C.$$

5.

$$\int \frac{x^4}{1-x^2}\ dx = \int -\frac{x^4}{x^2-1}\ dx = \int -\frac{x^4-1+1}{x^2-1}\ dx = \int -\left(x^2+1+\frac{1}{x^2-1}\right)\ dx$$
$$= \int -x^2 - 1 + \frac{1}{1-x^2}\ dx = \int -x^2 - 1 + \frac{1}{2}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)\ dx$$
$$= -\frac{1}{3}x^3 - x + \frac{1}{2}\log\left|\frac{1+x}{1-x}\right| + C$$

6.

$$\int \frac{\sqrt{1-x}}{1-\sqrt{x}}\ dx = \int \frac{\sqrt{1-\sin^2\theta}}{1-\sin\theta} 2\sin\theta\cos\theta\ d\theta = \int \frac{\cos^2\theta}{1-\sin\theta} 2\sin\theta\ d\theta$$
$$= \int (1+\sin\theta) 2\sin\theta\ d\theta = \int 2\sin\theta + 1 - \cos 2\theta\ d\theta = -2\cos\theta + \theta - \frac{1}{2}\sin 2\theta + C$$
$$= \theta - 2\cos\theta - \sin\theta\cos\theta + C = \sin^{-1}\sqrt{x} - 2\sqrt{1-x} - \sqrt{x}\sqrt{1-x} + C.$$

7.

$$\int e^{\cot x + 2\log \operatorname{cosec} x}\ dx = \int e^{\cot x}\operatorname{cosec}^2 x\ dx = \int -e^u du = -e^u + C = -e^{\cot x} + C.$$

8.

$$\int \frac{x e^{\sin^{-1} x}}{\sqrt{1-x^2}}\ dx = \int e^\theta \sin\theta\ d\theta = \frac{1}{2}(\sin\theta - \cos\theta)e^\theta + C$$
$$= \frac{1}{2}\left(x - \sqrt{1-x^2}\right) e^{\sin^{-1} x} + C$$

9.

$$\int \frac{3\sin x + 2\cos x}{2\sin x + 3\cos x}\ dx = \int \frac{\frac{12}{13}(2\sin x + 3\cos x) - \frac{5}{13}(2\cos x - 3\sin x)}{2\sin x + 3\cos x}\ dx$$
$$= \int \frac{12}{13} - \frac{5}{13}\frac{2\cos x - 3\sin x}{2\sin x + 3\cos x}\ dx = \frac{12}{13}x - \frac{5}{13}\log|2\sin x + 3\cos x| + C.$$

10.

$$\int \frac{1}{\sin x + \sin 2x}\ dx = \int \frac{1}{\sin x + 2\sin x \cos x}\ dx = \int \frac{1}{\sin x(1 + 2\cos x)}\ dx$$
$$= \int \frac{1}{(1-\cos^2 x)(1+2\cos x)} \sin x\ dx = \int -\frac{1}{(1-u^2)(1+2u)}\ du$$
$$= \int \frac{1}{2}\frac{1}{1+u} - \frac{1}{6}\frac{1}{1-u} - \frac{4}{3}\frac{1}{1+2u}\ du$$
$$= \frac{1}{2}\log|1+u| + \frac{1}{6}\log|1-u| - \frac{2}{3}\log|1+2u| + C$$
$$= \frac{1}{2}\log|1+\cos x| + \frac{1}{6}\log|1-\cos x| - \frac{2}{3}\log|1+2\cos x| + C$$
$$= \frac{1}{6}\log\left(\frac{(1+\cos x)^3(1-\cos x)}{(1+2\cos x)^4}\right) + C = \frac{1}{6}\log\left(\frac{\sin^2 x(1+\cos x)^2}{(1+2\cos x)^4}\right) + C$$
$$= \frac{1}{3}\log\left|\frac{\sin x(1+\cos x)}{(1+2\cos x)^2}\right| + C$$

Tie–breaker questions

1.

Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\ dx.$

Then

$$I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}\ dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\ dx,$$

$2I = \displaystyle\int_0^{\frac{\pi}{2}} 1\ dx = \frac{\pi}{2}$, so $I = \dfrac{\pi}{4}$.

2.

$$\int_1^{e^{\frac{1}{2}}} \frac{\sin^{-1} \log x}{x}\ dx = \int_0^{\frac{1}{2}} \sin^{-1} u\ du = \Big[u \sin^{-1} u\Big]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{u}{\sqrt{1-u^2}}\ du$$

$$= \frac{\pi}{12} + \Big[\sqrt{1-u^2}\Big]_0^{\frac{1}{2}} = \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1.$$

3.

$$\int_0^1 \frac{x^7 - 1}{\log x}\ dx = \int_0^1 \int_0^7 x^y\ dy\ dx = \int_0^7 \int_0^1 x^y\ dx\ dy = \int_0^7 \left[\frac{x^{y+1}}{y+1}\right]_0^1 dy$$

$$= \int_0^7 \frac{1}{y+1}\ dy = \Big[\log(y+1)\Big]_0^7 = \log 8.$$

Clearly not suitable for first–year students.

4.

Let $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^{\pi} x}\ dx.$

Then $I = \displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1 + \tan^{\pi}\left(\frac{\pi}{2} - x\right)}\ dx = \int_0^{\frac{\pi}{2}} \frac{1}{1 + \cot^{\pi} x}\ dx = \int_0^{\frac{\pi}{2}} \frac{\tan^{\pi} x}{\tan^{\pi} x + 1}\ dx,$

and $2I = \displaystyle\int_0^{\frac{\pi}{2}} 1\ dx = \frac{\pi}{2}, \quad I = \frac{\pi}{4}.$

5.

$$\int_0^{\frac{\pi}{6}} \sec^3 2\theta\ d\theta = \int_0^{\frac{\pi}{3}} \frac{1}{2} \sec^3 u\ du = \left[\frac{1}{4} \sec\theta \tan\theta + \frac{1}{4} \log|\sec\theta + \tan\theta|\right]_0^{\frac{\pi}{3}}$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{4} \log(2 + \sqrt{3}).$$