Simple proofs of Ramanujan's partition congruences

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Introduction

Let $p(n)$ be the number of partitions of n.
For example, $p(4) = 5$, since we can write

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1 \end{aligned}$$

and there are 5 such representations.
It was known to Euler that

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E(q)}$$

where

$$E(q) = (1-q)(1-q^2)(1-q^3)\ \cdots\ ,$$

and by convention, $p(0) = 1$.

Euler also knew that

$$\begin{aligned}E(q) &= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - - + +\ \cdots \\ &= \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}.\end{aligned}$$

It follows that

$$(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - - + +\ \cdots\ ) \times \sum_{n\geq 0} p(n)q^n = 1,$$

from which it follows that for $n > 0$,

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + + - -\ \cdots\ .$$

Thus,

$$\begin{aligned} p(1) &= p(0) = 1, \\ p(2) &= p(1) + p(0) = 2, \\ p(3) &= p(2) + p(1) = 3, \\ p(4) &= p(3) + p(2) = 5, \end{aligned}$$

and so on.

P. MacMahon, in 1917, calculated $p(n)$ for n up to 200, and presented his results thus:

1	7	42	176	627	1958	5604	⋯
1	11	56	231	792	2436	6842	⋯
2	15	77	297	1002	3010	8349	⋯
3	22	101	385	1255	3718	10143	⋯
5	30	135	190	1575	4565	12310	⋯

Look at the bottom row. Ramanujan did, and made the exciting discovery that

$$p(5n+4) \equiv 0 \pmod 5.$$

He also discovered that

$$p(7n+5) \equiv 0 \pmod 7 \quad \text{and} \quad p(11n+6) \equiv 0 \pmod{11}.$$

And these were just the simplest of his conjectures.
(There is nothing quite so simple for other primes.)
Ramanujan proved these three congruences, but his proof of the mod 11 congruence is much deeper than his proofs of the mod 5 and mod 7 congruences.
The purpose of this talk is to give simple proofs of all three congruences.
The proofs of the mod 5 and mod 7 congruences are simpler versions of Ramanujan's proofs, and the proof of the mod 11 congruence is new.

2 Two proofs of the mod 5 congruence

We start with two well–known identities, proofs of which can be found in Hardy & Wright, Introduction to the Theory of Numbers, for example.

Euler's identity

$$
\begin{aligned}
E &= (1-q)(1-q^2)(1-q^3)\ \cdots \\
&= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - - + +\ \cdots \\
&= \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2+n)/2}
\end{aligned}
$$

and Jacobi's identity

$$
\begin{aligned}
J = E^3 &= 1 - 3q + 5q^3 - 7q^6 + 9q^{10} - 11q^{15} + 13q^{21} \\
&\quad - 15q^{28} + 17q^{36} - 19q^{45} + -\ \cdots \\
&= \sum_{n \geq 0} (-1)^n (2n+1) q^{(n^2+n)/2}.
\end{aligned}
$$

First we notice that the exponents in the series for E, namely $0,\ 1,\ 2,\ 5,\ 7,\ 12,\ 15$ and so on are all congruent to 0, 1 or 2 modulo 5.
So we can write

$$E \equiv E_0 + E_1 + E_2,$$

where for $i = 0,\ 1,\ 2$, E_i comprises those terms in E in which the exponent is congruent to $i \pmod 5$.
Similarly, we notice that, modulo 5,

$$J \equiv J_0 + J_1,$$

where J_i comprises terms in which the exponent is congruent to $i \pmod 5$, since $(n^2 + n)/2 \equiv 0, 1$ or $3 \pmod 5$, but $2n + 1 \equiv 0 \pmod 5$ precisely when $(n^2 + n)/2 \equiv 3 \pmod 5$.

Also, it is easy to check that

$$(1-q)^5 \equiv 1-q^5 \pmod 5$$

so

$$\begin{aligned} E^5 &= (1-q)^5(1-q^2)^5(1-q^3)^5 \cdots \\ &\equiv (1-q^5)(1-q^{10})(1-q^{15}) \cdots \\ &= E(q^5). \end{aligned}$$

So now we have, modulo 5,

$$\begin{aligned} \sum_{n\geq 0} p(n)q^n = \frac{1}{E} = \frac{E^4}{E^5} \equiv \frac{E^4}{E(q^5)} \equiv \frac{EJ}{E(q^5)} \\ \equiv \frac{(E_0+E_1+E_2)(J_0+J_1)}{E(q^5)} \\ \equiv \frac{E_0J_0+E_0J_1+E_1J_0+E_1J_1+E_2J_0+E_2J_1}{E(q^5)}, \end{aligned}$$

and if we look for terms in which the exponent is congruent to 4 (mod 5), we find

$$\sum_{n \geq 0} p(5n+4)q^{5n+4} \equiv 0$$

and the mod 5 congruence is proved!
(This is a simplified version of Ramanujan's proof.)

An alternative proof using only J goes as follows. Modulo 5,

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E} = \frac{E^9}{E^{10}} = \frac{(E^3)^3}{(E^5)^2} \equiv \frac{J^3}{E(q^5)^2}$$
$$\equiv \frac{(J_0+J_1)^3}{E(q^5)^2}$$
$$\equiv \frac{J_0^3 + 3J_0^2 J_1 + 3J_0 J_1^2 + J_1^3}{E(q^5)^2},$$

so

$$\sum_{n\geq 0} p(5n+4)q^{5n+4} \equiv 0.$$

3 Proof of the mod 7 congruence

We have

$$E = E_0 + E_1 + E_2 + E_5,$$

where E_i comprises those terms of E in which the exponent is congruent to $i \pmod 7$,
and, modulo 7,

$$J \equiv J_0 + J_1 + J_3,$$

where J_i comprises those terms of J in which the exponent is congruent to $i \pmod 7$.
Note that $2n + 1 \equiv 0 \pmod 7$ whenever $(n^2 + n)/2 \equiv 6 \pmod 7$.

It follows that, modulo 7,

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E} = \frac{E^6}{E^7} = \frac{(E^3)^2}{E^7} = \frac{J^2}{E^7}$$
$$\equiv \frac{(J_0+J_1+J_3)^2}{E(q^7)}$$
$$= \frac{J_0^2+2J_0J_1+J_1^2+2J_0J_3+2J_1J_3+J_3^2}{E(q^7)},$$

So if we look for those terms in which the exponent is 5 (mod 7), we find

$$\sum_{n\geq 0} p(7n+5)q^{7n+5} \equiv 0.$$

4 Proof of the mod 11 congruence

We have

$$E = E_0 + E_1 + E_2 + E_5 + E_7 + E_{15}$$

and, modulo 11,

$$J \equiv J_0 + J_1 + J_3 + J_6 + J_{10},$$

since $2n + 1 \equiv 0 \pmod{11}$ whenever $(n^2 + n)/2 \equiv 4 \pmod{11}$.
So we have

$$\sum_{n \geq 0} p(n)q^n = \frac{1}{E} = \frac{E^{21}}{E^{22}} = \frac{(E^3)^7}{(E^{11})^2}$$

$$= \frac{J^7}{(E^{11})^2} \equiv \frac{J^7}{E(q^{11})^2}$$

$$\equiv \frac{(J_0 + J_1 + J_3 + J_6 + J_{10})^7}{E(q^{11})^2}.$$

If we now extract those terms in which the exponent is congruent to 6 (mod 11), we find

$$\sum_{n\geq 0} p(11n+6)q^{11n+6} \equiv \frac{P}{E(q^{11})^2},$$

where P is a polynomial in $J_0,\ J_1,\ J_3,\ J_6$ and J_{10} of degree 7.

Indeed, modulo 11,

$$
\begin{aligned}
P \equiv\ & 7J_0^6J_6 + 10J_0^5J_3^2 + J_0^4J_1J_6J_{10} + 8J_0^3J_1^3J_3 + 2J_0^3J_1J_3^2J_{10} \\
& + 8J_0^3J_6^3J_{10} + 3J_0^2J_1^2J_3J_6^2 + 3J_0^2J_1^2J_6J_{10}^2 + 3J_0^2J_3^2J_6^2J_{10} \\
& + 2J_0^2J_3J_6J_{10}^3 + 10J)0^2J_{10}^5 + 7J_0J_1^6 + J_0J_1^4J_3J_{10} + 2J_0J_1^2J_3^2J_6 \\
& + 3J_0J_1^2J_3^2J_{10}^2 + J_0J_1J_3J_6^4 + 2J_0J_1J_6^3J_{10}^2 + J_0J_3^4J_6J_{10} \\
& + 8J_0J_3^3J_{10}^3 + 10J_1^5J_6^2 + 2J_1^3J_3J_6^2J_{10} + 8J_1^3J_6J_{10}^3 + 10J_1^2J_3^5 \\
& + 8J_1J_3^3J_6^3 + 3J_1J_3^2J_6^2J_{10}^2 + J_1J_3J_6J_{10}^4 + 7J_1J_{10}^6 + 7J_3^6J_{10} \\
& + 7J_3J_6^6 + 10J_6^5J_{10}^2
\end{aligned}
$$

We want to show that $P \equiv 0$.

The crucial idea

We have

$$J^4 = E^{12} = E^{11}E \equiv E(q^{11})(E_0 + E_1 + E_2 + E_5 + E_7 + E_{15}).$$

That is,

$$(J_0+J_1+J_3+J_6+J_{10})^4 \equiv E(q^{11})(E_0+E_1+E_2+E_5+E_7+E_{15}).$$

If we extract those terms in which the exponents are $3,\ 6,\ 8,\ 9$ and $10 \pmod{11}$, we find

$$J_0^3J_3 + J_0J_1^3 + 6J_0J_1J_3J_6 + 7J_1^2J_6^2 + 3J_3J_6^2J_{10} + J_6J_{10}^3 \equiv 0,$$
$$J_0^3J_6 + 7J_0^2J_3^2 + 6J_0J_1J_6J_{10} + J_1^3J_3 + 3J_1J_3^2J_{10} + J_6^3J_{10} \equiv 0,$$
$$3J_0J_1^2J_6 + 6J_0J_3J_6J_{10} + J_0J_{10}^3 + 7J_1^2J_3^2 + J_1J_6^3 + J_3^3J_{10} \equiv 0,$$
$$3J_0^2J_3J_6 + 7J_0^2J_{10}^2 + J_0J_3^3 + 6J_1J_3J_6J_{10} + J_1^3J_6 + J_1J_{10}^3 \equiv 0,$$
$$J_0^3J_{10} + 6J_0J_1J_3J_6 + 3J_0J_1J_{10}^2 + J_1J_3^3 + J_3J_6^3 + 7J_6^2J_{10}^2 \equiv 0.$$

Now we make the observation that

$$
\begin{aligned}
P \equiv\ & (7J_1J_3J_{10} + 5J_0^2J_3 + 7J_1^3) \\
& \times (J_0^3J_3 + J_0J_1^3 + 6J_0J_1J_3J_6 + 7J_1^2J_6^2 + 3J_3J_6^2J_{10} + J_6J_{10}^3) \\
& + (7J_0^3 + 7J_0J_1J_{10} + 5J_6^2J_{10}) \\
& \times (J_0^3J_6 + 7J_0^2J_3^2 + 6J_0J_1J_6J_{10} + J_1^3J_3 + 3J_1J_3^2J_{10} + J_6^3J_{10}) \\
& + (7J_0J_3J_6 + 5J_0J_{10}^2 + 7J_3^3) \\
& \times (3J_0J_1^2J_6 + 6J_0J_3J_6J_{10} + J_0J_{10}^3 + 7J_1^2J_3^2 + J_1J_6^3 + J_3^3J_{10}) \\
& + (5J_1^2J_6 + 7J_3J_6J_{10} + 7J_{10}^3) \\
& \times (3J_0^2J_3J_6 + 7J_0^2J_{10}^2 + J_0J_3^3 + 6J_1J_3J_6J_{10} + J_1^3J_6 + J_1J_{10}^3) \\
& + (7J_0J_1J_6 + 5J_1J_3^2 + 7J_6^3) \\
& \times (J_0^3J_{10} + 6J_0J_1J_3J_6 + 3J_0J_1J_{10}^2 + J_1J_3^3 + J_3J_6^3 + 7J_6^2J_{10}^2)
\end{aligned}
$$

and so

$$P \equiv 0.$$