Some properties of the partitions of a number

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Introduction

In recent years, George E. Andrews introduced a new function, $spt(n)$, the smallest parts function, defined in terms of the partitions of n, and this function has proved to have some very interesting properties, as well as connections with other areas of investigation.

This talk arose out of my attempt to understand $spt(n)$.

I will discuss certain aspects of the partitions of a number at an elementary level, and incidentally shed some light on $spt(n)$.

We will begin by studying the number of 1's in the partitions of n.

We will then look at the number of 2's in those partitions of n that have no 1's.
From these investigations we will see, among other things, that $spt(n)$ satisfies

$$spt(n) \sim \left(\frac{\sqrt{6}}{\pi}\sqrt{n} + \frac{3}{\pi^2} \right) p(n) \text{ as } n \to \infty.$$

We will also examine the number of distinct parts in the partitions of n.
We uncover the remarkable fact that
the average number of distinct parts in the partitions of n
is exactly equal to
the average number of 1's in the partitions of n.

The number of 1's in the partitions of n

Of the $p(n)$ partitions of n, it is easy to see that there are $p(n-1)$ partitions with at least one 1:
simply strip a 1 off those partitions, and we obtain the partitions of $n-1$
and the process is reversible.

In the same way, we see that the number of partitions of n with at least two 1's is $p(n-2)$.
Strip a 1 off the $p(n-1)$ partitions with at least one 1.

Continuing in this way, we see that the number of partitions of n with at least k 1's is $p(n-k)$.

It follows that the number of partitions of n with exactly k 1's is $p(n-k) - p(n-k-1)$.

Let us illustrate the above with a table of the 15 partitions of 7.

$$
\begin{aligned}
7 &= 7 && \text{zero } 1's \quad p(7)-p(6)=4\\
&= 4+3\\
&= 5+2\\
&= 3+2+2\\
&= 6+1 && \text{one } 1 \quad p(6)-p(5)=4\\
&= 3+3+1\\
&= 4+2+1\\
&= 2+2+2+1\\
&= 5+1+1 && \text{two } 1's \quad p(5)-p(4)=2\\
&= 3+2+1+1\\
&= 4+1+1+1 && \text{three } 1's \quad p(4)-p(3)=2\\
&= 2+2+1+1+1\\
&= 3+1+1+1+1 && \text{four } 1's \quad p(3)-p(2)=1\\
&= 2+1+1+1+1+1 && \text{five } 1's \quad p(2)-p(1)=1\\
& && \text{six } 1's \quad p(1)-p(0)=0\\
&= 1+1+1+1+1+1+1 && \text{seven } 1's \quad p(0)=1
\end{aligned}
$$

We have that

The number of partitions of n with exactly k 1's is $p(n-k)-p(n-k-1)$.

It follows that the number of 1's in the partitions of n is

$$0\,(p(n)-p(n-1))+1\,(p(n-1)-p(n-2)) \\ +2\,(p(n-2)-p(n-3))+\ \cdots$$

$$=p(n-1)+p(n-2)+p(n-3)+\ \cdots\ .$$

For example, the number of 1's in the partitions of 7 is

$$p(6)+p(5)+p(4)+p(3)+p(2)+p(1)+p(0) \\ =11+7+5+3+2+1+1=30.$$

	partition	number of 1's
7	$= 7$	0
	$= 6 + 1$	1
	$= 5 + 2$	0
	$= 5 + 1 + 1$	2
	$= 4 + 3$	0
	$= 4 + 2 + 1$	1
	$= 4 + 1‘ + 1 + 1$	3
	$= 3 + 3 + 1$	1
	$= 3 + 2 + 2$	0
	$= 3 + 2 + 1 + 1$	2
	$= 3 + 1 + 1 + 1 + 1$	4
	$= 2 + 2 + 2 + 1$	1
	$= 2 + 2 + 1 + 1 + 1$	3
	$= 2 + 1 + 1 + 1 + 1 + 1$	5
	$= 1 + 1 + 1 + 1 + 1 + 1 + 1$	7
		total $= 30$

The number of 1's in the partitions of n is

$$p(n-1)+p(n-2)+p(n-3)+\ \cdots\ .$$

In order to approximate this, we make use of the approximation

$$p(n) \approx \frac{\exp\left\{K\sqrt{n}\right\}}{4n\sqrt{3}}\left(1-\left(\frac{1}{K}+\frac{K}{48}\right)\frac{1}{\sqrt{n}}\right).$$

where

$$K = \pi\sqrt{\frac{2}{3}}.$$

This follows from the Hardy–Ramanujan–Rademacher–Selberg formula for $p(n)$.

Using the trapezoidal rule,

$$p(n) + \cdots + p(0)$$
$$\approx \int_1^n \frac{\exp\left\{K\sqrt{x}\right\}}{4x\sqrt{3}} - \left(\frac{1}{K} + \frac{K}{48}\right)\frac{\exp\left\{K\sqrt{x}\right\}}{4x^{\frac{3}{2}}\sqrt{3}}\,dx + \frac{1}{2}p(n).$$

If in the denominator of the first term, we write $x = \sqrt{x} \cdot \sqrt{x}$, then perform one integration by parts, we find

$$p(n) + \cdots + p(0)$$
$$\approx \frac{\exp\left\{K\sqrt{n}\right\}}{2K\sqrt{3n}} - \frac{K}{192\sqrt{3}}\int_1^n \frac{1}{x} \cdot \frac{\exp\left\{K\sqrt{x}\right\}}{\sqrt{x}}\,dx + \frac{1}{2}p(n).$$

After a second integration by parts,

$$\begin{aligned} & p(n) + \cdots + p(0) \\ & \approx \frac{\exp\left\{K\sqrt{n}\right\}}{2K\sqrt{3n}} - \frac{\exp\left\{K\sqrt{n}\right\}}{96n\sqrt{3}} + \frac{1}{2}p(n) \\ & \approx \frac{\exp\left\{K\sqrt{n}\right\}}{2K\sqrt{3n}} + \frac{11\exp\left\{K\sqrt{n}\right\}}{96n\sqrt{3}} \end{aligned}$$

So the number of 1's in the partitions of n is

$$\begin{aligned} & p(n-1) + \cdots \ p(0) \\ & \approx \frac{\exp\left\{K\sqrt{n}\right\}}{2K\sqrt{3n}} - \frac{13\exp\left\{K\sqrt{n}\right\}}{96n\sqrt{3}} \\ & = \frac{\exp\left\{K\sqrt{n}\right\}}{2\pi\sqrt{2n}}\left(1 - \frac{13K}{48\sqrt{n}}\right). \end{aligned}$$

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The average number of 1's in the partitions of n is

$$\frac{p(n-1)+\ \cdots\ +p(0)}{p(n)}$$

$$\approx \frac{\sqrt{6n}}{\pi}\cdot\frac{1-\dfrac{13K}{48\sqrt{n}}}{1-\left(\dfrac{1}{K}+\dfrac{K}{48}\right)\dfrac{1}{\sqrt{n}}}$$

$$\approx \frac{\sqrt{6n}}{\pi}-\left(\frac{1}{2}-\frac{3}{\pi^2}\right).$$

n	$average$	$approx'n$	$diff'ce$
1000	24.46671877	24.46014131	0.0066
10000	77.77573135	77.77364367	0.0021

The smallest parts function, $spt(n)$

We saw that of the $p(n)$ partitions of n, $p(n-1)$ of them have smallest part 1, and the number of these smallest parts is $p(n-1) + \cdots + p(0)$.
We now examine the $p(n) - p(n-1)$ partitions of n that do not contain a 1.
Recall the case $n = 7$.

$$\begin{aligned} 7 &= 7 \\ &= 4+3 \\ &= 5+2 \\ &= 3+2+2 \end{aligned}$$

Of these four partitions of 7, two have smallest part 2, and the number of these smallest parts is three.
We will now count the number of 2's in those partitions of n with smallest part 2.

The generating function for partitions containing at least one 1 is

$$(q + q^2 + q^3 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots)\cdots$$

$$= \frac{q}{(1-q)(1-q^2)(1-q^3)\cdots}$$

If we want to count the number of 1's, the relevant generating function is

$$(q + 2q^2 + 3q^3 + \cdots)(1 + q^2 + q^4 + \cdots)(1 + q^3 + q^6 + \cdots)\cdots$$

$$= \frac{q}{(1-q)^2(1-q^2)(1-q^3)\cdots}$$

$$= \frac{1}{1-q} \cdot \sum_{n\geq 1} p(n-1)q^n,$$

from which it follows that the number of 1's in the partitions of n is

$$p(n-1) + \cdots + p(0),$$

as we saw below.
Similarly, the generating function for those partitions with smallest part 2 is

$$(q^2 + q^4 + q^6 + \cdots)(1 + q^3 + q^6 + \cdots)\cdots$$

while if we want to count the number of 2's, the relevant generating function is

$$\begin{aligned}(q^2 + 2q^4 + 3q^6 + \cdots)&(1 + q^3 + q^6 + \cdots) \\ &= \frac{q^2}{(1-q^2)^2(1-q^3)(1-q^4)\cdots} \\ &= \frac{1}{1+q}\cdot\frac{q^2}{(1-q)(1-q^2)(1-q^3)\cdots} \\ &= \frac{1}{1+q}\sum_{n\geq 2} p(n-2)q^n.\end{aligned}$$

we see that the number of 2's in those partitions with smallest part 2 is given by

$$p(n-2) - p(n-3) + p(n-4) - p(n-5) + - \cdots .$$

In the case $n = 7$, the number of 2's is

$$p(5)-p(4)+p(3)-p(2)+p(1)-p(0) = 7-5+3-2+1-1 = 3.$$

We will show that the number of 2's in those partitions of n with smallest part 2 is approximately and asymptotically

$$\frac{1}{2}p(n).$$

Clearly, this is not a good approximation at $n = 7$!

Let us suppose n is large, and define

$$r = \frac{p(n-1)}{p(n)}.$$

We have

$$p(n) \approx \frac{1}{4\sqrt{3}} \exp\left\{K\sqrt{n} - \log n - \left(\frac{1}{K} + \frac{K}{48}\right)\frac{1}{\sqrt{n}}\right\}.$$

It follows that

$$r = \exp\left\{-K(\sqrt{n} - \sqrt{n-1}) - (\log n - \log(n-1)) - \left(\frac{1}{K} + \frac{K}{48}\right)\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n-1}}\right)\right\}.$$

It is not hard to show that

$$r \approx \exp\left\{-\frac{K}{2\sqrt{n}} + \frac{1}{n} + \left(\frac{1}{2K} - \frac{11K}{96}\right)\frac{1}{n\sqrt{n}}\right\}.$$

Further,

$$\frac{p(n-k)}{p(n)} = \frac{p(n-1)}{p(n)} \frac{p(n-2)}{p(n-1)} \cdots \frac{p(n-k)}{p(n-k+1)}$$
$$\approx \exp\left\{ -\frac{K}{2}\left(\frac{1}{\sqrt{n}} + \cdots + \frac{1}{\sqrt{n-k+1}}\right) + \left(\frac{1}{n} + \cdots + \frac{1}{n-k+1}\right) + \left(\frac{1}{2K} - \frac{11K}{96}\right) \times \left(\frac{1}{n\sqrt{n}} + \cdots + \frac{1}{(n-k+1)\sqrt{n-k+1}}\right)\right\}$$

It is easily shown from here that

$$\frac{p(n-k)}{p(n)} \approx r^k \left(1 - \frac{K(k^2-k)}{8n\sqrt{n}}\right)$$

.

In other words,

$$\frac{p(n-k)}{p(n)} \approx r^k$$

provided k is not too large, say $k < n^{\frac{2}{3}}$.
To see this in action, consider $n = 1000$, $n^{\frac{2}{3}} = 100$,

$$r = \frac{p(999)}{p(1000)} \approx 0.9611983604,$$

$$r^{100} \approx 0.01911182376,$$

$$\frac{p(900)}{p(1000)} \approx 0.01728380345.$$

If $k < n^{\frac{2}{3}}$, the error in replacing $\frac{p(n-k)}{p(n)}$ by r^k is of the order of $\frac{1}{n^{\frac{1}{6}}}$, by which time

$$r^k \approx \exp\left\{-\frac{K}{2\sqrt{n}}n^{\frac{2}{3}}\right\} \approx \exp\left\{-\frac{Kn^{\frac{1}{6}}}{2}\right\} = o(1).$$

It follows that the number of 2's in the partitions with smallest part 2 is

$$
\begin{aligned}
&p(n-2) - p(n-3) + p(n-4) - p(n-5) + - \cdots \\
&\approx p(n)\left(r^2 - r^3 + r^4 - r^5 + - \cdots\right) \\
&= \frac{r^2}{1+r}p(n) \approx \frac{(1-\epsilon)^2}{1+(1-\epsilon)}p(n) \approx \frac{1-2\epsilon}{2-\epsilon}p(n) \\
&\approx \frac{1}{2}\left(1 - \frac{3\epsilon}{2}\right)p(n) \approx \frac{1}{2}\left(1 - \frac{3K}{4\sqrt{n}}\right)p(n).
\end{aligned}
$$

n	$(p(n-2) - + \cdots)/p(n)$	$approx'n$
1000	0.47101	0.4696
10000	0.490536	0.490381

In the same way, it can be shown that for each t, the number of t's in those partitions of n with smallest part t is approximately and asymptotically

$$\frac{(t-1)!}{t}\left(\frac{K}{2\sqrt{n}}\right)^{t-2}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right).$$

It follows that $spt(n)$, the number of smallest parts in the partitions of n is given by

$$\begin{aligned} spt(n) &\approx \left(\frac{\sqrt{6}}{\pi}\sqrt{n}-\left(\frac{1}{2}-\frac{3}{\pi^2}\right)\right)p(n)+\frac{1}{2}p(n) \\ &\approx \left(\frac{\sqrt{6}}{\pi}\sqrt{n}+\frac{3}{\pi^2}\right)p(n), \end{aligned}$$

as claimed in the introduction.

The number of distinct parts in the ptns of n

	partition	number of distinct parts
7	$= 7$	1
	$= 6 + 1$	2
	$= 5 + 2$	2
	$= 5 + 1 + 1$	2
	$= 4 + 3$	2
	$= 4 + 2 + 1$	3
	$= 4 + 1 + 1 + 1$	2
	$= 3 + 3 + 1$	2
	$= 3 + 2 + 2$	2
	$= 3 + 2 + 1 + 1$	3
	$= 3 + 1 + 1 + 1 + 1$	2
	$= 2 + 2 + 2 + 1$	2
	$= 2 + 2 + 1 + 1 + 1$	2
	$= 2 + 1 + 1 + 1 + 1 + 1$	2
	$= 1 + 1 + 1 + 1 + 1 + 1 + 1$	1

Let $p_{n,m}$ be the number of partitions of n with m distinct parts.

From the table on the previous slide, we see that $p_{7,1} = 2$, $p_{7,2} = 11$ and $p_{7,3} = 2$.

The average number of distinct parts in the partitions of 7 is

$$\frac{1 \times p_{7,1} + 2 \times p_{7,2} + 3 \times p_{7,3}}{p(7)} = \frac{1 \times 2 + 2 \times 11 + 3 \times 2}{15} = 2.$$

The generating function for the $p_{n,m}$ is

$$1 + \sum_{n\geq 1}\sum_{m\geq 1} p_{n,m} a^m q^n = \prod_{n\geq 1}\left(1 + a\left(q^n + q^{2n} + q^{3n} + \cdots\right)\right)$$

$$= \prod_{n\geq 1}\left(1 + \right.$$

$$= \prod_{n\geq 1} \frac{1 + (a}{1 -}$$

Note that if we set $a = 1$, this gives

$$1 + \sum_{n\geq 1}\sum_{m\geq 1} p_{n,m}q^n = \prod_{n\geq 1}\frac{1}{1-q^n} = \sum_{n\geq 0} p(n)q^n,$$

so

$$\sum_{m\geq 1} p_{n,m} = p(n),$$

as it should.

As a series in q,

$$\prod_{n\geq 1}\frac{1+(a-1)q^n}{1-q^n} = 1 + aq + 2aq^2 + (2a+a^2)q^3 + \cdots$$
$$+ (2a + 11a^2 + 2a^3)q^7 + \cdots ,$$

as expected.

We have

$$1+\sum_{n\geq 1}\sum_{m\geq 1} p_{n,m}a^m q^n = \prod_{n\geq 1}\frac{1+(a-1)q^n}{1-q^n}.$$

If we differentiate with respect to a, we find

$$\sum_{n\geq 1}\sum_{m\geq 1} m p_{n,m}a^{m-1} q^n = \prod_{n\geq 1}\frac{1+(a-1)q^n}{1-q^n}.\sum_{n\geq 1}\frac{q^n}{1+(a-1)q^n}.$$

If we now set $a=1$,

$$\begin{aligned}\sum_{n\geq 1}\sum_{m\geq 1} m p_{n,m} q^n &= \prod_{n\geq 1}\frac{1}{1-q^n}\cdot\sum_{n\geq 1} q^n \\ &= \frac{q}{1-q}\sum_{n\geq 0} p(n)q^n \\ &= \frac{1}{1-q}\sum_{n\geq 1} p(n-1)q^n,\end{aligned}$$

from which we deduce

$$\sum_{m\geq 1} mp_{n,m} = p(n-1) + \cdots + p(0),$$

and the average number of distinct parts in the partitions of n is

$$\frac{\sum_{m\geq 1} mp_{n,m}}{p(n)} = \frac{p(n-1) + \cdots + p(0)}{p(n)}$$

is also the average number of 1's in the partitions of n.

Concluding remarks

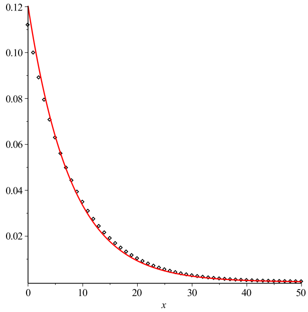
The number of 1's in the partitions of n is distributed, roughly speaking, as a negative exponential with mean

$$\mu \approx \frac{\sqrt{6}}{\pi}\sqrt{n} - \left(\frac{1}{2} - \frac{3}{\pi^2}\right),$$

while the number of distinct parts in the partitions of n is distributed, roughly speaking, normally, with mean μ and variance

$$\sigma^2 \approx \frac{3}{\pi^2}\left(\frac{\pi}{\sqrt{6}} - \frac{\sqrt{6}}{\pi}\right)\sqrt{n}.$$

For more detail, see my recent papers in the Fibonacci Quarterly.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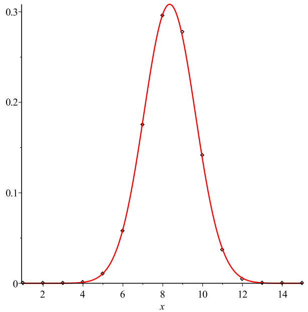
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Thank you,

and, Happy Birthday, George!