The power of q

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Part I

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An idea introduced by Euler.

"Number–busting"

How many ways can a number be written as a sum of (one or more) numbers?

For instance,

$$\begin{aligned} 4 &= 4 \\ &= 3+1 \\ &= 2+2 \\ &= 2+1+1 \\ &= 1+1+1+1. \end{aligned}$$

Note that $1+3$ is the same as $3+1$.

We usually write the parts in (weakly) decreasing order.

We say "the number of partitions of 4 is 5", and write

$$p(4) = 5.$$

Consider the product

$$\left(1 + q^1 + q^{1+1} + \ \cdots\ \right)\left(1 + q^2 + q^{2+2} + \ \cdots\ \right)\ \cdots$$

If we expand this product, a typical term might be

$$q^{1+1} \times q^2 \times 1 \times 1 \times \ \cdots\ = q^{1+1+2} = q^{2+1+1}$$

corresponding to the partition

$$2 + 1 + 1$$

of 4.

Thus we see that

$$\sum_{n\geq 0} p(n) q^n = \prod_{n\geq 1} \frac{1}{1-q^n} = \frac{1}{(1-q)(1-q^2)(1-q^3)\ \cdots}.$$

This is due to Euler.

We can use this product formula to obtain a recurrence for $p(n)$.

We have

$$\sum_{n\geq 0} p(n)q^n = P(q) = \frac{1}{(1-q)(1-q^2)(1-q^3)\ \cdots} = \prod_{n\geq 1} u_n,$$

where

$$u_n = (1-q^n)^{-1}.$$

If we differentiate w.r.to q, using the product rule, we find

$$\sum_{n\geq 1} np(n)q^{n-1} = P\cdot\sum_{n\geq 1}\frac{u_n'}{u_n}$$

$$= P\cdot\sum_{n\geq 1}\frac{-(1-q^n)^{-2}\cdot -nq^{n-1}}{(1-q^n)^{-1}}$$

$$\sum_{n\geq 1} np(n)q^{n-1} = P\cdot\sum_{n\geq 1}\frac{nq^{n-1}}{1-q^n},$$

or,

$$\begin{aligned}
\sum_{n\geq 1} np(n)q^{n} &= P\cdot\sum_{n\geq 1}\frac{nq^{n}}{1-q^n}\\
&= P\left(1\left(q+q^2+q^3+\cdots\right)\right.\\
&\quad +2\left(q^2+q^4+q^6+\cdots\right)\\
&\quad \left.+\cdots\right)\\
&= P\cdot\sum_{n\geq 1}\left(\sum_{d|n} d\right)q^n\\
&= P\cdot\sum_{n\geq 1}\sigma(n)q^n\\
&= \sum_{n\geq 1}\sigma(n)q^n\sum_{n\geq 0}p(n)q^n.
\end{aligned}$$

It follows that

$$\begin{aligned} np(n) &= p(n-1) + 2p(n-2) + 4p(n-3) + \cdots \\ &= \sum_{k=1}^{n} \sigma(k)p(n-k). \end{aligned}$$

Thus, suppose we know $p(0) = 1,\; p(1) = 1,\; p(2) = 2$ and $p(3) = 3$.
Then

$$\begin{aligned} 4p(4) &= \sigma(1)p(3) + \sigma(2)p(2) + \sigma(3)p(1) + \sigma(4)p(0) \\ &= 1 \times 3 + 3 \times 2 + 4 \times 1 + 7 \times 1 = 20, \end{aligned}$$

and

$$p(4) = 5.$$

In this way, we could calculate $p(n)$ indefinitely far.

Ramanujan's observations

We might, as Ramanujan did, observe some amazing facts that have motivated a great deal of research.

In 1917, P. A. MacMahon calculated $p(n)$ for $n \leq 200$, and tabulated the results in columns, thus:

1	7	42	176	627	$\cdots$
1	11	56	231	792	$\cdots$
2	15	77	297	1002	$\cdots$
3	22	101	385	1255	$\cdots$
5	30	135	490	1575	$\cdots$

Ramanujan noticed something that no one had noticed before, that the numbers at the foot of each column are divisible by 5. that is,

$$p(5n+4) \equiv 0 \pmod{5}.$$

Ramanujan's partition conjectures

He also noticed that

$$p(7n+5) \equiv 0 \pmod 7$$

and that

$$p(11n+6) \equiv 0 \pmod{11}.$$

Similarly simple congruences do not hold for any other prime!

We will prove all of these in the next hour or two.

Ramanujan further conjectured that

$$p(5^\alpha 7^\beta 11^\gamma n + \delta) \equiv 0 \pmod{5^\alpha 7^\beta 11^\gamma},$$

where δ is the reciprocal modulo $5^\alpha 7^\beta 11^\gamma$ of 24.

but this is not the case.

Ramanujan's partition congruences

It turns out that

$$p(5^\alpha 7^\beta 11^\gamma n + \delta) \equiv 0 \pmod{5^\alpha 7^{\beta'} 11^\gamma}$$

where for $\beta \geq 3$, $\beta' = \lfloor \dfrac{\beta+1}{2} \rfloor$.

This is really three separate congruences cobbled together.

The powers of 5 congruences were sort–of proved by Ramanujan (in his Lost Notebook, found in 1976) and by G. N. Watson, and more recently by Hirschhorn and Hunt,

the powers of 7 congruences by G.N. Watson, and more recently by F. G. Garvan,

and the powers of 11 congruences were proved by A. O. L. Atkin.

Before we develop the tools for proving the three basic congruences, let us look at some elementary partition theorems.

Euler discovered that the number, $p_d(n)$, of partitions of n into distinct parts (that is, each part has frequency of occurrence at most 1) is equal to the number, $p_o(n)$, of partitions of n into odd parts.

$$\begin{aligned}\sum_{n\geq 1} p_d(n)q^n &= \prod_{n\geq 1}(1+q^n) \\ &= \prod_{n\geq 1}\frac{1-q^{2n}}{1-q^n} \\ &= \frac{\prod_{n\geq 1}(1-q^{2n})}{\prod_{n\geq 1}(1-q^n)}\end{aligned}$$

Euler's theorem

$$\sum_{n\geq 1} p_d(n)q^n = \frac{\prod_{n\geq 1}(1-q^{2n})}{\prod_{n\geq 1}(1-q^{2n-1})(1-q^{2n})}$$

$$= \prod_{n\geq 1} \frac{1}{1-q^{2n-1}}$$

$$= \prod_{n\geq 1} \left(1 + q^{2n-1} + q^{2(2n-1)} + \cdots \right)$$

$$= \sum_{n\geq 0} p_o(n)q^n. \quad \blacksquare$$

It is not hard to give an "arithmetic" proof of Euler's theorem.

More elementary partition theorems

You can create your own partition theorems.

For instance, the number of partitions, $p_{\{\leq 2\}}(n)$ of n where no part occurs more than twice, is given by

$$\begin{aligned}\sum_{n\geq 0} p_{\{\leq 2\}}(n)q^n &= \prod_{n\geq 1}\left(1+q^n+q^{n+n}\right)\\ &= \prod_{n\geq 1}\frac{1-q^{3n}}{1-q^n}\\ &= \prod_{n\geq 1}\frac{(1-q^{3n})}{(1-q^{3n-2})(1-q^{3n-1})(1-q^{3n})}\\ &= \prod_{n\geq 1}\frac{1}{(1-q^{3n-2})(1-q^{3n-1})}\\ &= \sum_{n\geq 0} p_{\{1,2;3\}}(n)q^n,\end{aligned}$$

$p_{\{1,2;3\}}(n)$ denotes partitions with parts not divisible by 3.

$n = 6$

$p_{\{\leq 2\}}(6)$	$p_{\{1,2;3\}}(6)$
6	$5+1$
$5+1$	$4+2$
$4+2$	$4+1+1$
$4+1+1$	$2+2+2$
$3+3$	$2+2+1+1$
$3+2+1$	$2+1+1+1+1$
$2+2+1+1$	$1+1+1+1+1+1$

let $p^*(n)$ denote the number of partitions of n where no part occurs precisely once.

Then

$$
\begin{aligned}
\sum_{n\geq 0} p^*(n)q^n &= \prod_{n\geq 1}\left(1+q^{2n}+q^{3n}+\ \cdots\ \right)\\
&= \prod_{n\geq 1}\left(1+\frac{q^{2n}}{1-q^n}\right)\\
&= \prod_{n\geq 1}\frac{1-q^n+q^{2n}}{1-q^n}\\
&= \prod_{n\geq 1}\frac{1+q^{3n}}{1-q^{2n}}\\
&= \prod_{n\geq 1}\frac{(1-q^{6n})}{(1-q^{2n})(1-q^{3n})}
\end{aligned}
$$

$$\sum_{n\geq 0} p^*(n)q^n$$

$$= \prod_{n\geq 1} \frac{(1-q^{6n})}{(1-q^{6n-4})(1-q^{6n-2})(1-q^{6n})(1-q^{6n-3})(1-q^{6n})}$$

$$= \prod_{n\geq 1} \frac{1}{(1-q^{6n-4})(1-q^{6n-3})(1-q^{6n-2})(1-q^{6n})}$$

$$= \sum_{n\geq 0} p_{\{2,3,4,6;6\}}(n)q^n.$$

$n = 8$

$p^*(n)$	$p_{\{2,3,4,6;6\}}(n)$
$4+4$	8
$3+3+1+1$	$6+2$
$2+2+2+1+1$	$4+4$
$2+2+1+1+1+1$	$3+3+2$
$1+1+1+1+1+1+1+1$	$2+2+2+2$

Just one more example.

If $p^{**}(n)$ denotes the number of partitions of n in which each part occurs with frequency 2, 3, 4, 5, 6, 7 or 9, then

$$p^{**}(n) = p_{\{2,3,4,6,9,10,12,14,15,18,20,21,22;24\}}(n).$$

I leave the proof as an exercise!

Euler

Consider the infinite product

$$(1+a)(1+aq)(1+aq^2)\ \cdots\ = \sum_{n\geq 0} c_n(q)a^n.$$

Note that $c_0 = 1$.

If we put aq for a, we get

$$(1+aq)(1+aq^2)(1+aq^3)\ \cdots\ = \sum_{n\geq 0} q^n c_n(q)a^n.$$

It follows that

$$\begin{aligned}\sum_{n\geq 0} c_n a^n &= (1+a)\sum_{n\geq 0} q^n c_n a^n \\ &= \sum_{n\geq 0} q^n c_n a^n + \sum_{n\geq 1} q^{n-1} c_{n-1} a^n.\end{aligned}$$

It follows that for $n \geq 1$,

$$c_n = q^n c_n + q^{n-1} c_{n-1},$$

$$(1 - q^n)c_n = q^{n-1} c_{n-1},$$

$$c_n = \frac{q^{n-1}}{1 - q^n} c_{n-1},$$

and by induction, for $n \geq 1$,

$$c_n = \frac{q^{(n^2-n)/2}}{(1-q)(1-q^2)\cdots(1-q^n)}.$$

So we have Euler's result

$$(1+a)(1+aq)(1+aq^2)\cdots = 1 + \sum_{n\geq 1} \frac{q^{(n^2-n)/2}}{(1-q)\cdots(1-q^n)} a^n.$$

Interlude

There is a neat finite version of Euler's identity.

We need some notation.

Define

$$(q)_n = (1-q)(1-q^2)\cdots(1-q^n) \text{ for } n \geq 1, \quad (q)_0 = 1.$$

Define the q–binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_{(q)}$ for $0 \leq k \leq n$ by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q)} = \frac{(q)_n}{(q)_k (q)_{n-k}}.$$

Then for $n \geq 1$ and $0 \leq k \leq n$,

$$\begin{bmatrix} n \\ k \end{bmatrix}_{(q)} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{(q)} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_{(q)}.$$

We obtain a q–Pascal's triangle,

$$\begin{array}{ccccc} 1 & & & & \\ 1 & 1 & & & \\ 1 & 1+q & 1 & & \\ 1 & 1+q+q^2 & 1+q+q^2 & 1 & \\ 1 & \cdots & \cdots & \cdots & 1 \end{array}$$

It is now obvious that $\begin{bmatrix} n \\ k \end{bmatrix}_{(q)}$ is a polynomial.

It is easy to show by induction that

$$(1+a)(1+aq)\cdots(1+aq^{n-1}) = \sum_{k=0}^{n} q^{(k^2-k)/2} \begin{bmatrix} n \\ k \end{bmatrix}_{(q)} a^k.$$

This is the "q–binomial theorem".

Back to the infinite version

We have

$$(1+a)(1+aq)(1+aq^2)\cdots = \sum_{k\geq 0} \frac{q^{(k^2-k)/2}}{(q)_k} a^k.$$

If we replace q by q^2 and replace a by aq, we obtain

$$(1+aq)(1+aq^3)(1+aq^5)\cdots = \sum_{k\geq 0} \frac{q^{k^2}}{(q^2)_k} a^k.$$

We now do something tricky. Replace a by aq^{-2n}. We obtain

$$(1+aq^{-(2n-1)})\cdots(1+aq^{-1})(1+aq)(1+aq^3)\cdots = \sum_{k\geq 0} \frac{q^{k^2-2nk}}{(q^2)_k} a^k.$$

Now multiply by $a^{-n}q^{n^2} = a^{-1}q^{2n-1} \cdot a^{-1}q^{2n-3} \cdots a^{-1}q$, and we obtain

$$(1+a^{-1}q)(1+a^{-1}q^3)\cdots(1+a^{-1}q^{2n-1}) \\ \times(1+aq)(1+aq^3)(1+aq^5)\cdots \\ = \sum_{k\geq 0} \frac{q^{k^2-2nk+n^2}}{(q^2)_k} a^{k-n} \\ = \sum_{k\geq -n} \frac{q^{k^2}}{(q^2)_{n+k}} a^k.$$

(This can, alternatively, be proved by induction!)

If we now let $n \to \infty$, we obtain

$$\prod_{k\geq 1}(1+a^{-1}q^{2k-1})(1+aq^{2k-1}) = \sum_{-\infty}^{\infty} \frac{q^{k^2}}{(q^2)_\infty} a^k = \frac{1}{(q^2)_\infty}\sum_{-\infty}^{\infty} a^k q^{k^2}.$$

.

That is,

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^k q^{k^2}.$$

This is Jacobi's celebrated Triple Product Identity.

JTP can be used to expand products as series, but it can also be used to sum series of the form

$$\sum_{-\infty}^{\infty} q^{ak^2+bk} \quad \text{or} \quad \sum_{-\infty}^{\infty}(-1)^k q^{ak^2+bk}$$

with $a \geq b$ to products.

Again, there is a neat finite version,

$$\prod_{k=1}^{m}(1+a^{-1}q^{2k-1})\prod_{k=1}^{n}(1+aq^{2k-1}) = \sum_{k=-m}^{n} a^k q^{k^2} \begin{bmatrix} m+n \\ m+k \end{bmatrix}_{(q^2)}.$$

We had

$$\prod_{n\geq 1}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^k q^{k^2}.$$

If we put $-aq$ for q, then replace q^2 by q, then "fold" the sum over, we find

$$\begin{aligned}(1-a^{-1})\prod_{n\geq 1}(1-a^{-1}q^n)(1-aq^n)(1-q^n) \\ = \quad 1+\sum_{k\geq 1}(-1)^k\left(a^k - a^{-k-1}\right)q^{(k^2+k)/2}.\end{aligned}$$

If we now divide by $(1-a^{-1})$, we obtain

$$\begin{aligned}\prod_{n\geq 1}(1-a^{-1}q^n)(1-aq^n)(1-q^n) \\ = \quad 1+\sum_{k\geq 1}(-1)^k\left(a^k + a^{k-1} + \cdots + a^{-k}\right)q^{(k^2+k)/2}.\end{aligned}$$

If we let $a \to 1$ in this, we obtain a celebrated result of Jacobi, which I may refer to as “Jacobi's cube”,

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

Another form of Jacobi's triple product identity,

$$\prod_{n\geq 1}(1-a^{-1}q^{3n-2})(1-aq^{3n-1})(1-q^{3n})$$

$$= \sum_{-\infty}^{\infty}(-1)^k a^k q^{(3k^2+k)/2}.$$

Euler's “pentagonal number theorem” is the case $a=1$.

$$\prod_{n\geq 1}(1-q^n) = \sum_{-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}.$$

Thus,

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{1-q-q^2+q^5+q^7-q^{12}-q^{15}+\ \cdots}$$

This gives us another recurrence for $p(n)$, For $n \geq 1$,

$$p(n) = p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\cdots .$$

Incidentally, F. Franklin gave a beautiful combinatorial proof of Euler's pentagonal number theorem making use of Ferrers' graph.

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}$$

$$= \sum_{-\infty}^{\infty}(4n+1)q^{2n^2+n}$$

$$= \left[\frac{d}{da}\left(\sum_{-\infty}^{\infty} a^{4n+1}q^{2n^2+n}\right)\right]_{a=1}$$

$$= \left[\frac{d}{da}\left(a\prod_{n\geq 1}(1+a^{-4}q^{4n-3})(1+a^4q^{4n-1})(1-q^{4n})\right)\right]_{a=1}$$

$$= \left[\prod_{n\geq 1}(1+a^{-4}q^{4n-3})(1+a^4q^{4n-1})(1-q^{4n})\right.$$

$$\left.\times\left(1+a\sum_{n\geq 1}\left(-\frac{4a^{-5}q^{4n-3}}{1+a^{-4}q^{4n-3}}+\frac{4a^3q^{4n-1}}{1+a^4q^{4n-1}}\right)\right)\right]_{a=1}$$

$$\prod_{n\geq 1}(1-q^n)^3$$
$$=\prod_{n\geq 1}(1+q^{4n-3})(1+q^{4n-1})(1-q^{4n})$$
$$\times\left(1-4\sum_{n\geq 1}\left(\frac{q^{4n-3}}{1+q^{4n-3}}-\frac{q^{4n-1}}{1+q^{4n-1}}\right)\right).$$

Now, the product

$$\prod_{n\geq 1}(1+q^{4n-3})(1+q^{4n-1}(1-q^{4n})$$
$$=\prod_{n\geq 1}(1+q^{2n-1})(1-q^{4n})$$
$$=\prod_{n\geq 1}\frac{(1+q^n)(1-q^{4n})}{1+q^{2n}}$$

$$= \prod_{n\geq 1} \frac{(1-q^{2n})}{(1-q^n)} \frac{(1-q^{2n})}{(1-q^{4n})}(1-q^{4n})$$

$$= \prod_{n\geq 1} \frac{(1-q^{2n})^2}{(1-q^{4n})}.$$

So we have

$$\prod_{n\geq 1}(1-q^n)^3 = \prod_{n\geq 1} \frac{(1-q^{2n})^2}{(1-q^n)} \times \left(1 - 4\sum_{n\geq 1}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right)\right),$$

$$\prod_{n\geq 1} \frac{(1-q^n)^4}{(1-q^{2n})^2} = 1 - 4\sum_{n\geq 1}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right).$$

Now

$$\sum_{-\infty}^{\infty}(-1)^n q^{n^2} = \prod_{n\geq 1}(1-q^{2n-1})^2(1-q^{2n})$$
$$= \prod_{n\geq 1}(1-q^{2n-1})(1-q^n) = \prod_{n\geq 1}\frac{(1-q^n)^2}{(1-q^{2n})},$$

so we have

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^2 = 1 - 4\sum_{n\geq 1}\left(\frac{q^{4n-3}}{1+q^{4n-3}} - \frac{q^{4n-1}}{1+q^{4n-1}}\right).$$

Put $-q$ for q, and we obtain

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = 1 + 4\sum_{n\geq 1}\left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}}\right).$$

Jacobi's two–squares theorem

$$\left(\sum_{-\infty}^{\infty} q^{n^2}\right)^2 = 1+4\sum_{n\geq 1}\left(\sum_{\substack{d|n\\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n\\ d\equiv -1 \pmod 4}} 1\right) q^n.$$

So we have Jacobi's two–squares theorem,
For $n \geq 1$, the number of representations of n as a sum of two squares is

$$r_2(n) = 4\left(d_{\{1;4\}} - d_{\{-1;4\}}(n)\right).$$

If $n = 2^\alpha p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k} q_1^{\beta_1} \cdots q_l^{\beta_l}$, where the p_i are primes $\equiv 1 \pmod 4$ and the q_j are primes $\equiv -1 \pmod 4$, and $P > 0$ is arbitrary, then

$$\sum_{\substack{d|n\\ d\equiv 1 \pmod 4}} d^P - \sum_{\substack{d|n\\ d\equiv -1 \pmod 4}} d^p =$$

$$= \prod_{i=1}^{k} \left(1 + p_i^P + \cdots + p_i^{\alpha_i P}\right)$$

$$\times \prod_{j=1}^{l} \left(1 - q_j^P + - \cdots + (-1)^{\beta_j} q^{\beta_j P}\right)$$

$$= \prod_{i=1}^{k} \frac{p_i^{(\alpha_i+1)P} - 1}{p_i^P - 1} \prod_{j=1}^{l} \frac{1 + (-1)^{\beta_j} q^{(\beta_j+1)P}}{1 + q_j^P}.$$

If we let $P \to 0$, we find

$$\sum_{\substack{d|n \\ d\equiv 1 \pmod 4}} 1 - \sum_{\substack{d|n \\ d\equiv -1 \pmod 4}} 1 = \prod_{i=1}^{k} (\alpha_i + 1) \prod_{j=1}^{l} \frac{1 + (-1)^{\beta_j}}{2},$$

$$r_2(n) = 4 \prod_{i=1}^{k} (\alpha_i + 1) \prod_{j=1}^{l} \frac{1 + (-1)^{\beta_j}}{2}.$$

$$r_2(n) = 0$$

if any β_j is odd, that is, if the squarefree part of n is divisible by a prime $\equiv -1 \pmod 4$, while if all β_j are even, $r_2(n) = \prod_{i=1}^{k}(\alpha_1 + 1)$.

In particular,

A number is the sum of two squares if and only if its squarefree part is not divisible by a prime congruent to -1 modulo 4.

We can write

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2} = \frac{1}{2}\sum_{-\infty}^{\infty}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

It follows that

$$\prod_{n\geq 1}(1-q^n)^6$$

$$= \frac{1}{4}\sum_{m,n=-\infty}^{\infty}(-1)^{m+n}(2m+1)(2n+1)q^{(m^2+m+n^2+n)/2}$$

$$= \frac{1}{4}\left(\sum_{m+n\ \text{even}}(2m+1)(2n+1)q^{(m^2+m+n^2+n)/2} - \sum_{m+n\ \text{odd}}(2m+1)(2n+1)q^{(m^2+m+n^2+n)/2}\right).$$

In the first sum, write $r = (m+n)/2,\ s = (m-n)/2$. Then $m = r+s,\ n = r-s$.
In the second sum, write $r = (m-n-1)/2,\ s = (m+n+1)/2$. Then $m = r+s,\ n = s-r-1$.

$$\prod_{n\geq 1}(1-q^n)^6 = \frac{1}{4}\left(\sum_{r,s=-\infty}^{\infty}\left((2r+1)^2-(2s)^2\right)q^{r^2+r+s^2}\right.$$
$$\left. -\sum_{r,s=-\infty}^{\infty}\left((2s)^2-(2r+1)^2\right)q^{r^2+r+s^2}\right)$$
$$= \frac{1}{2}\left(\sum_{-\infty}^{\infty}(2r+1)^2 q^{r^2+r}\sum_{-\infty}^{\infty}q^{s^2} - \sum_{-\infty}^{\infty}q^{r^2+r}\sum_{-\infty}^{\infty}(2s)^2 q^{s^2}\right)$$
$$= \sum_{r\geq 0}(2r+1)^2 q^{r^2+r}\sum_{-\infty}^{\infty}q^{s^2} - \sum_{r\geq 0}q^{r^2+r}\sum_{-\infty}^{\infty}(2s)^2 q^{s^2}$$

$$
\begin{aligned}
\prod_{n\geq 1}(1-q^n)^6 &= \prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n}) \\
&\quad \times \left(1+4q\frac{d}{dq}\right)\left(\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})\right) \\
&- \prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n}) \\
&\quad \times 4\frac{d}{dq}\left(\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})\right)
\end{aligned}
$$

$$
\begin{aligned}
&\prod_{n\geq 1}(1-q^n)^6 \\
= \quad &\prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})(1+q^{2n})^2(1-q^{2n}) \\
&\times\left(1+4\sum_{n\geq 1}\left(\frac{2(2n)q^{2n}}{1+q^{2n}}-\frac{2nq^{2n}}{1-q^{2n}}\right)\right) \\
&-\prod_{n\geq 1}(1+q^{2n})^2(1-q^{2n})(1+q^{2n-1})^2(1-q^{2n}) \\
&\times 4\sum_{n\geq 1}\left(\frac{2(2n-1)q^{2n-1}}{1+q^{2n-1}}-\frac{2nq^{2n}}{1-q^{2n}}\right)
\end{aligned}
$$

$$\prod_{n\geq 1}(1-q^n)^6$$
$$= \prod_{n\geq 1}\frac{(1-q^{2n})^4}{(1-q^n)^2}\left(1+8\sum_{n\geq 1}\left(\frac{2nq^{2n}}{1+q^{2n}}-\frac{(2n-1)q^{2n-1}}{1+q^{2n-1}}\right)\right)$$

$$\prod_{n\geq 1}\frac{(1-q^n)^8}{(1-q^{2n})^4}=1+8\sum_{n\geq 1}(-1)^n\frac{nq^n}{1+q^n},$$

$$\left(\sum_{-\infty}^{\infty}(-1)^n q^{n^2}\right)^4=1+8\sum_{n\geq 1}(-1)^n\frac{nq^n}{1+q^n},$$

$$\left(\sum_{n\geq 0} q^{n^2}\right)^4 = 1 + 8\sum_{n\geq 1} \frac{nq^n}{1+(-1)^n q^n},$$

$$= 1 + 8\sum_{n\geq 1} \frac{nq^n}{1-q^n} - 8\sum_{n \text{ even}} nq^n \left(\frac{1}{1-q^n} - \frac{1}{1+q^n}\right)$$

$$= 1 + 8\sum_{n\geq 1} \frac{nq^n}{1-q^n} - 8\sum_{n \text{ even}} \frac{2nq^{2n}}{1-q^{2n}}$$

$$= 1 + 8\sum_{4\nmid n} \frac{nq^n}{1-q^n}$$

$$= 1 + 8\sum_{n\geq 1} \left(\sum_{d|n,\ 4\nmid n} d\right) q^n.$$

Jacobi's four–squares theorem

So we have Jacobi's four–squares theorem, for $n \geq 1$,

$$r_4(n) = 8 \sum_{d|n,\ 4\nmid d} d = 8 \sum_{d|n,\ d\equiv 1,2 \text{ or } 3 \pmod 4} d.$$

In particular, $r_4(n) > 0$.

Every number is the sum of four squares.