The power of q

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Part II

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Ramanujan's partition congruences

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

Note that the power on q, $(n^2+n)/2 \equiv 0,\ 1 \text{ or } 3 \mod 5$.

And when $(n^2+n)/2 \equiv 3 \pmod 5$, $n \equiv 2 \pmod 5$, and then the coefficient, $(2n+1) \equiv 0 \pmod 5$.

So we can write

$$\prod_{n\geq 1}(1-q^n)^3 \equiv J_0 + J_1 \pmod 5,$$

where J_0 consists of those terms in which the power of q is congruent to 0 modulo 5, and J_1 those terms in which the power of q is congruent to 1 modulo 5.

Also, $(1-q)^5 \equiv 1-q^5 \pmod 5$.

So we can write

$$\begin{aligned}
\sum_{n\geq 0} p(n)q^n &= \prod_{n\geq 1} \frac{1}{1-q^n} \\
&= \frac{\prod_{n\geq 1}(1-q^n)^9}{\prod_{n\geq 1}(1-q^n)^{10}} \\
&= \frac{\left(\prod_{n\geq 1}(1-q^n)^3\right)^3}{\left(\prod_{n\geq 1}(1-q^n)^5\right)^2} \\
&\equiv \frac{(J_0+J_1)^3}{\left(\prod(1-q^{5n})\right)^2} \\
&= \frac{J_0^3+3J_0^2J_1+3J_0J_1^2+J_1^3}{\left(\prod_{n\geq 1}(1-q^{5n})\right)^2}.
\end{aligned}$$

There are no terms on the right in which the power of q is 4 modulo 5, so

$$\sum_{n\geq 0} p(5n+4)q^n \equiv 0 \pmod 5. \quad \blacksquare$$

$$\prod_{n\geq 1}(1-q^n)^3 = \sum_{n\geq 0}(-1)^n(2n+1)q^{(n^2+n)/2}.$$

The power, $(n^2+n)/2 \equiv 0,\ 1,\ 3$ or $6 \pmod 7$, and when $(n^2+n)/2 \equiv 6 \pmod 7$, $n \equiv 3 \pmod 7$, and then the coefficient $(2n+1) \equiv 0 \pmod 7$.

So we can write

$$\prod_{n\geq 0}(1-q^n)^3 \equiv J_0 + J_1 + J_3,$$

where J_i consists of those terms in which the power of q is $\equiv i \pmod 7$, $i = 0,\ 1,\ 3$.

Also, $(1-q)^7 \equiv 1 - q^7 \pmod 7$.

So

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \prod_{n\geq 1}\frac{1}{1-q^n}\\ &= \frac{\left(\prod_{n\geq 1}(1-q^n)^3\right)^2}{\prod_{n\geq 1}(1-q^n)^7}\\ &\equiv \frac{(J_0+J_1+J_3)^2}{\prod_{n\geq 1}(1-q^{7n})}\\ &\equiv \frac{J_0^2+2J_0J_1+J_1^2+2J_0J_3+2J_1J_3+J_3^2}{\prod_{n\geq 1}(1-q^{7n})}.\end{aligned}$$

There are no terms in which the power of q is $\equiv 5 \pmod 7$, so

$$\sum_{n\geq 0} p(7n+5)q^n \equiv 0 \pmod 7. \quad \blacksquare$$

In the same way, we can write

$$\prod_{n\geq 1}(1-q^n)^3 \equiv J_0 + J_1 + J_3 + J_6 + J_{10} \pmod{11},$$

where J_i consists of those terms in which the power of q is $\equiv i \pmod{11}$.

$$\begin{aligned}\sum_{n\geq 0} p(n)q^n &= \frac{\left(\prod_{n\geq 1}(1-q^n)^3\right)^7}{\left(\prod_{n\geq 1}(1-q^n)^{11}\right)^2} \\ &\equiv \frac{(J_0+J_1+J_3+J_6+J_{10})^7}{\left(\prod_{n\geq 1}(1-q^{11n})\right)^2}.\end{aligned}$$

It follows that, modulo 11,

$$\sum_{n\geq 0} p(11n+6)q^n \equiv \frac{P_6}{\prod_{n\geq 1}(1-q^n)^2}$$

where P is a polynomial in the J_i homogeneous of degree 7 in the J_i, of 30 terms. Not obviously congruent to 0 mod 11.

$$P_6 = 7J_0^6 J_6 + 10 J_0^5 J_3^2 + J_0^4 J_1 J_6 J_{10} + \cdots + 7 J_6^5 J_{10}^2.$$

Here is the trick I discovered last year.

$$(J_0 + J_1 + J_3 + J_6 + J_{10})^4 \equiv \left(\prod_{n\geq 1}(1-q^n)^3\right)^4$$

$$= \left(\prod_{n\geq 1}(1-q^n)\right)^{12}$$

$$= \left(\prod_{n\geq 1}(1-q^n)\right)^{11} \cdot \prod_{n\geq 1}(1-q^n)$$

$$\equiv \prod_{n\geq 1}(1-q^{11n}) \cdot \sum_{-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}$$

$$(J_0 + J_1 + J_3 + J_6 + J_{10})^4 \equiv \prod_{n\geq 1}(1-q^{11n})\,(E_0 + E_1 + E_2 + E_4 + E_5 + E_7)\,.$$

where each E_i consists of powers $\equiv i \pmod{11}$.
Comparing terms in which the power of q is $3,\ 6,\ 8,\ 9$ and $10 \pmod{11}$ gives five quartics in the J_i that are $\equiv 0 \pmod{11}$
For example, $Q_3 =$

$$J_0^3 J_3 + J_0 J_1^3 + 6J_0 J_1 J_3 J_{10} + 7J_1^2 J_6^2 + 3J_3 J_6^2 J_{10} + J_6 J_{10}^3 \equiv 0.$$

If we can express P_6 as a "linear" combination of these five quartics (using cubics as coefficients) then we will have succeeded in proving that

$$\sum_{n\geq 0} p(11n+6)q^n \equiv 0 \pmod{11}.$$

And, indeed we can!

$$P_6 \equiv C_3 Q_3 + C_0 Q_6 + C_9 Q_8 + C_8 Q_9 + C_7 Q_{10},$$

where, for example,

$$C_3 = 7 J_9 J_3 J_{10} + 5 J_0^2 J_3 + 7 J_1^3,$$

and the other C_i are of similar form.

See my paper in the Journal of Number Theory 139 (2014), 205–209, for details.

Some notation and some simple results

We can say a lot more about the preceding results, but it may help to introduce some more notation.

$$(a;q)_\infty = \prod_{k\geq 0}(1-aq^k),$$

$$(a_1,a_2,\ \cdots\ a_n;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_n;q)_\infty,$$

$$\left(\begin{matrix} a_1,a_2,\ \cdots\ ,a_n \\ b_1,b_2,\ \cdots\ ,b_m \end{matrix};q\right)_\infty = \frac{(a_1,a_2,\ \cdots\ ,a_n;q)_\infty}{(b_1,b_2,\ \cdots\ ,b_m;q)_\infty}.$$

Euler's product has many notations, among them

$$E(q) = \prod_{n\geq 1}(1-q^n) = (q;q)_\infty.$$

Bruce Berndt persuaded me to use $(q^k;q^k)_\infty$ in preference to $(q)_\infty$, so I often use that in my papers. .

Ramanujan defined

$$\phi(q) = \sum_{-\infty}^{\infty} q^{n^2} = \prod_{n\geq 1}(1+q^{2n-1})^2(1-q^{2n})$$

$$= \quad \frac{(q^2;q^2)_\infty^5}{(q;q)_\infty^2(q^4;q^4)_\infty^2} = \frac{E(q^2)^5}{E(q)^2E(q^4)^2},$$

$$\phi(-q) = \frac{E(q)^2}{E(q^2)},$$

$$\psi(q) = \sum_{n\geq 0} q^{(n^2+n)/2} = \frac{E(q^2)^2}{E(q)},$$

$$\psi(-q) = \frac{E(q)E(q^4)}{E(q^2)^2}$$

Jacobi's triple product identity in one form becomes

$$(-a^{-1}q, -aq, q^2; q^2)_\infty = \sum_{-\infty}^{\infty} a^n q^{n^2},$$

in another,

$$(a^{-1}q, aq^2, q^3; q^3)_\infty = \sum_{-\infty}^{\infty} (-1)^k a^k q^{(3k^2+k)/2},$$

and yet another,

$$\begin{aligned}&(a^{-1}q, aq, q; q)_\infty \\ &= 1 + \sum_{k\geq 1} \left(a^k + a^{k-1} + \ \cdot \ + a^{-k}\right) q^{(k^2+k)/2}.\end{aligned}$$

Ramanujan's most beautiful identity

We saw that, modulo 5,

$$E(q) = E_0 + E_1 + E_2.$$

We can do much better.

We had

$$(a^{-1}q, aq, q; q)_\infty = 1 + \sum_{k \geq 1} \left(a^k + \cdots + a^{-k}\right) q^{(k^2+k)/2}.$$

Let $\eta = \exp\{\frac{2\pi i}{5}\}$. We set $a = \eta$ in the above, and use the fact that

$$\eta^k + \cdots + \eta^{-k} = \begin{cases} 1 & \text{if } k \equiv 0 \pmod 5, \\ -(\eta^2 + \eta^{-2}) & \text{if } k \equiv 1 \pmod 5, \\ 0 & \text{if } k \equiv 2 \pmod 5, \\ \eta^2 + \eta^{-2} & \text{if } k \equiv 3 \pmod 5, \\ -1 & \text{if } k \equiv 4 \pmod 5. \end{cases}$$

We now split the sum in five, according to the residue modulo 5 of k

For $k \equiv 0$, write $k = 5m,\ m \geq 1$, for $k \equiv 1$ write $k = 5m+1,\ m \geq 0$, for $k \equiv 2$, write $k = 5m+2,\ m \geq 0$, for $k \equiv 3$ write $k = -5m-2,\ m \leq -1$ and for $k \equiv 4$, write $k = -5m-1,\ m \leq -1$.
We obtain

$$\begin{aligned}
&(\eta^{-1}q, \eta q, q; q)_\infty \\
&= \sum_{-\infty}^{\infty} q^{(25m^2+5m)/2} + (\eta^2+\eta^{-2})q \sum_{-\infty}^{\infty} q^{(25m^2+5m)/2} \\
&= (q^{10}, q^{15}, q^{25}; q^{25})_\infty + (\eta^2+\eta^{-2})q(q^5, q^{20}, q^{25}; q^{25})_\infty,
\end{aligned}$$

where we have used JTP to sum the series.

We found that

$$
\begin{aligned}
& (\eta q, \eta^{-1} q, q; q)_\infty \\
= \; & (q^{10}, q^{15}, q^{25}; q^{25})_\infty + (\eta^2 + \eta^{-2}) q (q^5, q^{20}, q^{25}; q^{25})_\infty .
\end{aligned}
$$

Taking $a = \eta^2$ gives

$$
\begin{aligned}
& (\eta^2 q, \eta^{-2} q, q; q)_\infty \\
= \; & (q^{10}, q^{15}, q^{25}; q^{25})_\infty + (\eta + \eta^{-1}) q (q^5, q^{20}, q^{25}; q^{25})_\infty .
\end{aligned}
$$

If we multiply these two equations, we find

$$
\begin{aligned}
& (q; q)_\infty (q^5; q^5)_\infty = \\
= \; & (q^{10}, q^{15}, q^{25}; q^{25})_\infty^2 - q (q^5; q^5)_\infty (q^{25}; q^{25})_\infty \\
& \qquad - q^2 (q^5, q^{20}, q^{25}; q^{25})_\infty^2 .
\end{aligned}
$$

If we divide by $(q^5;q^5)_\infty$, we find

$$(q;q)_\infty = (q^{25};q^{25})_\infty \times \left(\left(\begin{matrix} q^{10}, q^{15} \\ q^5, q^{20} \end{matrix} ; q^{25} \right)_\infty - q \; - q^2 \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix} \; q^{25} \right)_\infty \right).$$

We now define

$$R = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix} ; q^5 \right)_\infty ,$$

and then we have

$$E(q) = E(q^{25}) \left(R(q^5)^{-1} - q - q^2 R(q^5) \right),$$

which is the refined form of

$$E = E_0 + E_1 + E_2.$$

We have

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{E} = \frac{E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)}{E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)}.$$

We can write the denominator in two different ways.

$$\begin{aligned} & E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) \\ = \; & E(q^{25})^5 \left(R(q^5)^{-1} - q - q^2 R(q^5)\right) \\ & \times \left(R^{-1}(q^5) - \eta q - \eta^2 q^2 R(q^5)\right) \\ & \cdots \times \left(R(q^5) - \eta^4 q - \eta^3 q^2 R(q^5)\right) \\ = \; & E(q^{25})^5 \left(R(q^5)^{-5} - 11q^5 - q^{10} R(q^5)^5\right). \end{aligned}$$

Alternatively,

$$
\begin{aligned}
& E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) \\
= & \prod_{n\geq 1}(1-q^n)(1-\eta^n q^n)(1-\eta^{2n}q^n)(1-\eta^{3n}q^n)(1-\eta^{4n}q^n) \\
= & \prod_{5|n}(1-q^{5n}) \cdot \prod_{5\nmid n}(1-q^{5n}) \\
= & \prod_{n\geq 1}(1-q^{5n})^6 / \prod_{n\geq 1}(1-q^{25n}) = \frac{E(q^5)^6}{E(q^{25})}.
\end{aligned}
$$

And the numerator is

$$
\begin{aligned}
& E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q) = E(q^{25})^4 \\
& \times \left(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1}\right. \\
& \left. +5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right).
\end{aligned}
$$

It follows that

$$\sum_{n\geq 0} p(n)q^n = \frac{E(q^{25})^5}{E(q^5)^6}$$
$$\times \left(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1}\right.$$
$$\left.+5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right).$$

And therefore **Fanfare!**

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6}.$$

Hardy thought this perhaps the best, in some sense (beauty, simplicity, elegance) of Ramanujan's work.

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The corrsponding formula for the prime 7 is

$$\sum_{n\geq 0} p(7n+5)q^n = 7\frac{E(q^7)^3}{E(q)^4} + 49q\frac{E(q^7)^7}{E(q)^8}.$$

Ramanujan's congruences for powers of 5

We saw that

$$\sum_{n\geq 0} p(5n+4)q^n = 5\frac{E(q^5)^5}{E(q)^6}$$

and that

$$\frac{1}{E(q)} = \frac{E(q^{25})^5}{E(q^5)^6}$$
$$\times \left(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1}\right.$$
$$\left. +5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\right).$$

If we substitute the second equation into the first, we find

$$\sum_{n\geq 0} p(5n+4)q^n$$

$$= 5\frac{E(q^{25})^{30}}{E(q^5)^{31}} \times \Big(R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2R(q^5)^{-2} + 3q^3R(q^5)^{-1} + 5q^4 - 3q^5R(q^5) + 2q^6R(q^5)^2 - q^7R(q^5)^3 + q^8R(q^5)^4\Big)^6$$

$$= 5\frac{E(q^{25})^{30}}{E(q^5)^{31}}\Big(R(q^5)^{-24} + 6qR(q^5)^{-23} + \cdots \cdots + q^{48}R(q^5)^{24}\Big).$$

If we now extract those terms in which the power of q is congruent to 4 modulo 5, divide by q^4 and replace q^5 by q, we find

$$\sum_{n\geq 0} p(25n+24)q^n = 5\frac{E(q^5)^{30}}{E(q)^{31}}$$
$$\times\left(315R(q)^{-20} + 18640qR(q)^{-15} + 139305q^2R(q)^{-10}\right.$$
$$+127020q^3R(q)^{-5} + 106425q^4 - 127020q^5R(q)^5$$
$$\left.+139305q^6R(q)^{10} - 186040q^7R(q)^{15} + 315q^8R(q)^{20}\right).$$

Now, we saw before that

$$E(q^{25})^5\left(R(q^5)^{-5} - 11q^5 - q^{10}R(q^5)^5\right) = \frac{E(q^5)^6}{E(q^{25})},$$

so

$$R(q)^{-5} - 11q - q^2R(q)^5 = \frac{E(q)^6}{E(q^5)^6}.$$

Using the facts that

$$\begin{aligned}(1-11x-x^2)^2 &= 1-22x+119x^2+22x^3+x^4,\\ (1-11x-x^2)^3 &= 1-33x+360x^2-1263x^3-360x^4\\ &\quad -33x^5-x^6\\ (1-11x-x^2)^4 &= 1-44x+722x^2-5192x^3+13195x^4\\ &\quad +5192x^5+722x^6+44x^7+x^8\end{aligned}$$

It is routine to show that

$$\begin{aligned}&\sum_{n\geq 0} p(25n+24)q^n\\ = \quad & 25\frac{E(q^5)^{30}}{E(q)^{31}}\Big(63\left(R(q)^{-5}-11q-R(q)^5\right)^4\\ & +52\times 5q\left(R(q)^{-5}-11q-q^2R(q)^5\right)^3\\ & +63\times 5^5q^2\left(R(q)^{-5}-11q-q^2R(q)^5\right)^2\\ & +6\times 5^8q^3\left(R(q)^{-5}-11q-q^2R(q)^5\right)+5^{10}q^4\Big).\end{aligned}$$

That is,

$$\sum_{n\geq 0} p(25n+24)q^n = 25\frac{E(q^5)^{30}}{E(q)^{31}}$$

$$\times \left(63\left(\frac{E(q)^6}{E(q^5)^6}\right)^4 +52\times 5q\left(\frac{E(q)^6}{E(q^5)^6}\right)^3\right.$$

$$\left. +63\times 5^5q^2\left(\frac{E(q)^6}{E(q^5)^6}\right)^2 +6\times 5^8q^3\left(\frac{E(q)^6}{E(q^5)^6}\right) + 5^{10}q^4\right)$$

$$= 25\left(63\frac{E(q^5)^6}{E(q)^7} + 52\times 5q\frac{E(q^5)^{12}}{E(q)^{13}}\right.$$

$$+63\times 5^5q^2\frac{E(q^5)^{18}}{E(q)^{19}} + 6\times 5^8q^3\frac{E(q^5)^{24}}{E(q)^{25}}$$

$$\left. +5^{10}q^4\frac{E(q^5)^{30}}{E(q)^{31}}\right).$$

This process can be repeated indefinitely, and a general theorem obtained.

$$\sum_{n\geq 0} p(5^\alpha n + \delta_\alpha) q^n = \begin{cases} \sum_{i\geq 1} x_{\alpha,i} q^{i-1} \dfrac{E(q^5)^{6i-1}}{E(q)^{6i}}, & \alpha \text{ odd} \\ \sum_{i\geq 1} x_{\alpha,i} q^{i-1} \dfrac{E(q^5)^{6i}}{E(q)^{6i+1}}, & \alpha \text{ even} \end{cases}$$

where

$$\nu_5(x_{\alpha,i}) \geq \begin{cases} \alpha + \lfloor \frac{1}{2}(5i-5) \rfloor & \alpha \text{ odd,} \\ \alpha + \lfloor \frac{1}{2}(5i-4) \rfloor & \alpha \text{ even} \end{cases}$$

and

$$\delta_\alpha = \begin{cases} \frac{1}{24}(19 \times 5^\alpha + 1) & \alpha \text{ odd,} \\ \frac{1}{24}(23 \times 5^\alpha + 1) & \alpha \text{ even.} \end{cases}$$

Note that $1 \leq \delta_\alpha < 5^\alpha$ and $24\delta_\alpha \equiv 1 \pmod{5^\alpha}$.

Details may be found in M. D. Hirschhorn and D. C. Hunt, "A simple proof of the Ramanujan conjectures for powers of 5", J. Reine Angew. Math. 326(1981), 1–17.

You may also like to consult F. G. Garvan, "A simple proof of Watson's partition congruences for powers of 7", J. Austral. Math. Soc. (Ser. A) 36(1984), 316–334.'

Returning to the basic congruences, there is more we can say.

$$\text{Modulo 5,} \quad E(q)^3 = J_0 + J_1 + J_3,$$

$$\begin{aligned}
J_0 &= 1 + 9q^{10} - 11q^{15} - 19q^{45} + 21q^{55} + \cdots \\
&\equiv 1 - q^{10} - q^{15} + q^{45} + q^{55} - \cdots \\
&= \sum (-1)^n q^{(25n^2-5n)/2} \\
&= (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \\
J_1 &= -3q - 7q^6 - 13q^{21} + 17q^{36} - 23q^{66} - 27q^{91} + \cdots \\
&\equiv -3q + 3q^6 + 3q^{21} - 3q^{36} - 3q^{66} + 3q^{91} + \cdots \\
&= -3q \sum (-1)^n q^{(25n^2-15n)/2} \\
&= -3q(q^5, q^{20}, q^{25}; q^{25})_\infty, \\
J_3 &= 5q^3 - 15q^{28} + 25q^{78} - 35q^{153} + - \cdots \\
&= 5q^3 \left(1 - 3q^{25} + 5q^{75} - \cdots\right) = 5q^3 E(q^{25})^3.
\end{aligned}$$

$$\sum_{n\geq 0} p(n)q^n \equiv \frac{(J_0+J_1)^3}{E(q^5)^2}$$

$$\equiv \frac{(q^{10},q^{15},q^{25};q^{25})_\infty - 3q(q^5,q^{20},q^{25};q^{25})_\infty)^3}{E(q^5)^2}$$

It follows that

$$\sum_{n\geq 0} p(5n)q^n \equiv \frac{(q^2,q^2,q^2,q^3,q^3,q^3,q^5,q^5,q^5;q^5)_\infty}{E(q)^2},$$

$$\sum_{n\geq 0} p(5n+1)q^n \equiv \frac{(q,q^2,q^2,q^3,q^3,q^4,q^5,q^5,q^5;q^5)_\infty}{E(q)^2},$$

$$\sum_{n\geq 0} p(5n+2)q^n \equiv 2\frac{(q,q,q^2,q^3,q^4,q^4,q^5,q^5,q^5;q^5)_\infty}{E(q)^2},$$

$$\sum_{n\geq 0} p(5n+3)q^n \equiv 3\frac{(q,q,q,q^4,q^4,q^4,q^5,q^5,q^5;q^5)_\infty}{E(q)^2}.$$

Similarly, modulo 7,

$$
\begin{aligned}
\sum_{n\ge0} p(7n)q^n &\equiv \frac{(q^3,q^3,q^4,q^4,q^7,q^7;q^7)_\infty}{E(q)},\\
\sum_{n\ge0} p(7n+1)q^n &\equiv \frac{(q^2,q^3,q^4,q^5,q^7,q^7;q^7)_\infty}{E(q)},\\
\sum_{n\ge0} p(7n+2)q^n &\equiv 2\frac{(q^2,q^2,q^5,q^5,q^7,q^7;q^7)_\infty}{E(q)},\\
\sum_{n\ge0} p(7n+3)q^n &\equiv 3\frac{(q,q^3,q^4,q^6,q^7,q^7;q^7)_\infty}{E(q)},\\
\sum_{n\ge0} p(7n+4)q^n &\equiv 5\frac{(q,q^2,q^5,q^6,q^7,q^7;q^7)_\infty}{E(q)},\\
\sum_{n\ge0} p(7n+6)q^n &\equiv 11\frac{(q,q,q^6,q^6,q^7,q^7;q^7)_\infty}{E(q)}.
\end{aligned}
$$

There are similar simple relations modulo 11 for $\sum_{n\geq 0} p(11n+r)q^n$.

For example,

$$\sum_{n\geq 0} p(11n)q^n \equiv \frac{(q^2,q^3,q^4,q^5,q^6,q^7,q^8,q^9,q^{11},q^{11};q^{11})_\infty}{E(q)}.$$

My proofs follow from a formula for $E(q)^{10}$, which itself follows from Winquist's identity.

I refer you to
M.D. Hirschhorn, "A generalisation of Winquist's identity and a conjecture of Ramanujan", J. Indian Math. Soc., 51(1987), 49–55,
and to
M. D. Hirschhorn, "Winquist and the Atkin–Swinnerton-Dyer partition congruences for modulus 11", Australas. J. Combin., 22(2000), 101–104.