The power of q

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Part III

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The Rogers-Ramanujan identities

are

$$\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty},$$

$$\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

They are equivalent to

$$(q;q)_\infty\sum_{n\geq 0}\frac{q^{n^2}}{(q;q)_n}=(q^2,q^3,q^5;q^5)_\infty,$$

$$(q;q)_\infty\sum_{n\geq 0}\frac{q^{n^2+n}}{(q;q)_n}=(q,q^4,q^5;q^5)_\infty.$$

They are the case $a = 1$ of the identities

$$(aq;q)_\infty \sum_{k\geq 0} \frac{a^k q^{k^2}}{(q;q)_k} = \sum_{k\geq 0} (-1)^k a^{2k} q^{(5k^2+k)/2} (1 - a^2 q^{4k+2}) \frac{(aq;q)_k}{(q;q)_k}$$

and

$$(aq;q)_\infty \sum_{k\geq 0} \frac{a^k q^{k^2+k}}{(q;q)_k} = \sum_{k\geq 0} (-1)^k a^{2k} q^{(5k^2+3k)/2} (1 - a q^{2k+1}) \frac{(aq;q)_k}{(q;q)_k}.$$

Ramanujan's proof, which may or may not come from Rogers' proof, goes as follows:

Let

$$G(a) = 1 + \sum_{k\geq 1} (-1)^k a^{2k} (1 - aq^{2k}) \frac{(aq;q)_{k-1}}{(q;q)_k}.$$

Then it can be shown that

$$G(a) = (1 - aq)G(aq) + aq(1 - aq)(1 - aq^2)G(aq^2).$$

If we now let

$$G(a) = (aq;q)_\infty F(a),$$

then

$$F(a) = F(aq) + aqF(aq^2).$$

It is then easy to show that

$$F(a) = \sum_{k\geq 0} \frac{a^k q^{k^2}}{(q;q)_k},$$

and to obtain our two identities.

For details, please consult M. D. Hirschhorn, "An identity that may have changed the course of history", Contemporary Mathematics, to appear (paper185 on my website)

We can also deduce that

$$\cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{1+\cfrac{aq^3}{1+\cdots}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{a^n q^{n^2+n}}{(q;q)_n}}{\displaystyle\sum_{n\geq 0} \frac{a^n q^{n^2}}{(q;q)_n}}$$

For $a = 1$,

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cdots}}}} = \frac{\displaystyle\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n}}{\displaystyle\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n}} = \begin{pmatrix} q, q^4 \\ q^2, q^3 \end{pmatrix}_{\infty} ; q^5 = R.$$

Suppose

$$R = \sum_{n\geq 0} u_n q^n, \quad R^{-1} = \sum_{n\geq 0} v_n q^n.$$

We can calculate the $\{u_n\}$ and $\{v_n\}$ in two different ways. We can differentiate, and obtain

$$\begin{aligned}
&\sum_{n\geq 1} n u_n q^n \\
= \quad & -\sum_{n\geq 0}\left(\frac{(5n+1)q^{5n+1}}{1-q^{5n+1}} - \frac{(5n+2)q^{5n+2}}{1-q^{5n+2}}\right. \\
&\left. -\frac{(5n+3)q^{5n+3}}{1-q^{5n+3}} + \frac{(5n+4)q^{5n+4}}{1-q^{5n+4}}\right)\sum_{n\geq 0} u_n q^n,
\end{aligned}$$

So if we define

$$d^*(n) = \sum_{\substack{d|n \\ d\equiv\pm 1 \pmod 5}} d - \sum_{\substack{d|n \\ d\equiv\pm 2 \pmod 5}} d$$

we have, for $n \geq 1$,

$$nu_n = -\sum_{k\geq 1} d^*(k)u_{n-k}$$

and, similarly, for $n \geq 1$,

$$v_n = \sum_{k\geq 1} d^*(k)v_{n-k}.$$

This leads to the values

$$u_n = 1, -1, 1, 0, -1|1, -1, 1, 0, -1|2, -3, 2, 0.-2|4, -4, 3, -1, -3, \ cdots$$

and

$$v_n = 1, 1, 0, -1, 0|1, 1, -1, -2, 0|2, 2, -1, -3, -1|3, 3, -2, -5, -1| \cdots$$

George Szekeres, in 1968 or 69 noticed that both sequences exhibit periodicity of sign, with period 5.

Szekeres and I managed to find the dominant term in each case by using Cauchy's integral formula, but proving the error small had to wait for L. B. Richmond.

$$u_n \sim \frac{\sqrt{2}}{(5n)^{\frac{3}{4}}} \exp\left\{\frac{4\pi}{25}\sqrt{5n}\right\} \times \left(\cos\left(\frac{4\pi n}{5} + \frac{3\pi}{25}\right) + O\left(n^{-\frac{1}{2}}\right)\right)$$

$$v_n \sim \frac{\sqrt{2}}{(5n)^{\frac{3}{4}}} \exp\left\{\frac{4\pi}{25}\sqrt{5n}\right\} \times \left(\cos\left(\frac{2\pi n}{5} + \frac{4\pi}{25}\right) + O\left(n^{-\frac{1}{2}}\right)\right)$$

These formulae show that Szekeres' observation is eventually true.

But we can prove it true (with a handful of early exceptions).

We need the quintuple product identity.

For a thorough recent survey, see S. Cooper, "The quintuple product identity", International Journal of Number Theory 02 115(2005)

$$\left(\begin{matrix} a^{-2}, a^2q, q \\ a^{-1}, aq \end{matrix}; q\right)_\infty$$
$$= \sum_{-\infty}^{\infty}(-1)^n a^{3n} q^{(3n^2+n)/2} + \sum_{-\infty}^{\infty}(-1)^n a^{3n-1} q^{(3n^2-n)/2}.$$

Useful for expanding the product on the left, as we shall see, but also useful for summing two related triple products.

For example,

$$E(q) = \sum_{-\infty}^{\infty}(-1)^n q^{(3n^2+n)/2}$$

$$= \left(\sum_{-\infty}^{\infty}(-1)^m q^{(75m^2+5m)/2} - q^5 \sum_{-\infty}^{\infty}(-1)^m q^{(75m^2-55m)/2}\right)$$

$$-q\sum_{-\infty}^{\infty}(-1)^m q^{(75m^2-25m)/2}$$

$$-q^2\left(\sum_{-\infty}^{\infty}(-1)^m q^{(75m^2+35m)/2} - q^5 \sum_{-\infty}^{\infty}(-1)^m q^{(75m^2+65m)/2}\right)$$

$$= \left(\begin{matrix} q^{10}, q^{15}, q^{25} \\ q^5, q^{20} \end{matrix}; q^{25}\right)_\infty - q(q^{25}; q^{25})_\infty$$

$$-q^2\left(\begin{matrix} q^5, q^{20}, q^{25} \\ q^{10}, q^{15} \end{matrix}; q^{25}\right)_\infty$$

$$= E(q^{25})\left(R(q^5)^{-1} - q - q^2 R(q^5)\right).$$

Proof of QPI

$$\begin{aligned}
&\left(\begin{matrix} a^{-2}, a^2q, q \\ a^{-1}, aq \end{matrix} ; q\right)_\infty \\
= &\ \frac{(-a^{-1}, -aq, q; q)_\infty (a^{-2}q, a^2q, q^2; q^2)_\infty}{(q^2;q^2)_\infty} \\
= &\ \frac{1}{(q^2;q^2)_\infty} \sum_{k,l=-\infty}^{\infty} (-1)^l a^{k+2l} q^{(k^2+k)/2+l^2}.
\end{aligned}$$

We split the sum into three, according to the residue of $k+2l$ modulo 3.

If $k+2l \equiv 0$, let $k+2l = 3m$, $t = m-l$, then $k = m+2t$, $l = m-t$ and the corresponding sum is

$$\begin{aligned}
&\sum_{m,t=-\infty}^{\infty} (-1)^{m+t} a^{3m} q^{(3m^2+m)/2+3t^2+t} \\
= &\ (q^2;q^2)_\infty (a^{-3}q, a^3q^2, q^3; q^3)_\infty.
\end{aligned}$$

The sum corresponding to $k + 2l \equiv -1 \pmod 3$ is

$$\sum_{m,t=-\infty}^{\infty} (-1)^{m+t} a^{3m-1} q^{(3m^2-m)/2+3t^2+t} = a^{-1}(q^2;q^2)_\infty (a^{-3}q^2, a^3 q, q^3; q^3)_\infty,$$

while the sum corresponding to $k + 2l \equiv 1 \pmod 3$ is

$$\sum_{m,t=-\infty}^{\infty} (-1)^{m+t} a^{3m+1} q^{(3m^2+3m)/2+3t^2+3t+1} = 0.$$

Returning to R, we have

$$\begin{aligned}
\sum_{n\geq 0} u_n q^n &= \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty \\
&= \frac{1}{(q^5;q^5)_\infty} \left(\begin{matrix} q, q^4, q^5 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty \\
&= \frac{1}{(q^5;q^5)_\infty} \left(\sum_{-\infty}^{\infty} (-1)^n q^{(15n^2+7n)/2} \right. \\
&\qquad \left. - \sum_{-\infty}^{\infty} (-1)^n q^{(15n^2+13n+2)/2}\right).
\end{aligned}$$

We now split each of the two sums into five, according to the residue modulo 5 of n.
This gives ten sums.
We then pair them off into five pairs, according to the residue modulo 5 of the quadratic exponent.
Remarkably, each pair can be summed by the quintuple product identity into a single product.

We arrive at

$$\sum_{n\geq 0} u_n q^n = \frac{1}{(q^5;q^5)_\infty}\Bigg(\left(\begin{matrix} q^{30}, q^{95}, q^{125} \\ q^{15}, q^{110} \end{matrix} ; q^{125}\right)_\infty - q\left(\begin{matrix} q^{20}, q^{105}, q^{125} \\ q^{10}, q^{115} \end{matrix} ; q^{125}\right)_\infty + q^2\left(\begin{matrix} q^{55}, q^{70}, q^{125} \\ q^{35}, q^{90} \end{matrix} ; q^{125}\right)_\infty - q^{18}\left(\begin{matrix} q^{5}, q^{120}, q^{125} \\ q^{60}, q^{65} \end{matrix} ; q^{125}\right)_\infty - q^4\left(\begin{matrix} q^{45}, q^{80}, q^{125} \\ q^{40}, q^{85} \end{matrix} ; q^{125}\right)_\infty\Bigg).$$

This gives rise to five identities, typical of which is

$$\sum_{n\geq 0} u_{5n}q^n = 1/(q,q^2,q^3,q^3,q^4,q^5,q^7,q^8,q^9,q^{10},q^{11},q^{12},$$
$$q^{13},q^{14},q^{15},q^{17},q^{18},q^{20},q^{21},q^{22},q^{22},q^{23},q^{24};q^{25})_\infty.$$

There are five identities of the same sort for the v_n, as well. It is then easy to show that

$$u_3 = u_8 = u_{13} = u_{23} = 0, \quad v_2 = v_4 = v_9 = 0,$$

and otherwise,

$$u_{5n} > 0,\ u_{5n+1} < 0,\ u_{5n+2} > 0,\ u_{5n+3} < 0,\ u_{5n+4} < 0,$$
$$v_{5n} > 0,\ v_{5n+1} > 0,\ v_{5n+2} < 0,\ v_{5n+3} < 0,\ v_{5n+4} < 0.$$

For details, see M. D. Hirschhorn, "On the expansion of Ramanujan's continued fraction, and its reciprocal", Ramanujan Journal 2(1998), 521–527.
See also M. D. Hirschhorn, "On the 2– and 4–dissections of Ramanujan's continued fraction and its reciprocal", Ramanujan Journal 24(2011), 85–92.

Remarkably, Ramanujan calculated u_n and v_n beyond 1000 (!) (LNB), but left us no conjectures.
I forgot to detail the second way to calculate the u_n and v_n.
We have

$$\sum_{n\geq 0} u_n q^n = \begin{pmatrix} q, q^4 \\ q^2, q^3 \end{pmatrix}; q^5 \Big)_\infty$$

$$= \frac{(q, q^4, q^5; q^5)_\infty}{(q^2, q^3, q^5; q^5)_\infty}$$

$$= \frac{\sum_{-\infty}^{\infty} (-1)^k q^{(5k^2-3k)/2}}{\sum_{-\infty}^{\infty} (-1)^k q^{(5k^2-k)/2}}$$

$$= \frac{1 + \sum_{k\geq 1} (-1)^k \left(q^{(5k^2-3k)/2} + q^{(5k^2+3k)/2} \right)}{1 + \sum_{k\geq 1} (-1)^k \left(q^{(5k^2-k)/2} + q^{(5k^2+k)/2} \right)}$$

We obtain the recurrence, for $n \geq 0$,

$$u_n + \sum_{k\geq 1} (-1)^k \left(u_{n-(5k^2-k)/2} + u_{n-(5k^2+k)/2} \right) = 0,$$

unless n is of the form $(5l^2 \pm 3l)/2$, in which case the right side is not 0, but $(-1)^l$.
Similarly,

$$v_n + \sum_{k\geq 1} (-1)^k \left(v_{n-(5k^2-3k)/2} + v_{n-(5k^3+3k)/2} \right) = 0$$

unless n is of the form $(5l^2 \pm l)/2$, in which case the right hand side is $(-1)^l$

There is a finite version of Ramanujan's continued fraction.

$$\cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{1+\cfrac{aq^3}{1+\ddots\cfrac{aq^{n-1}}{1+\cfrac{aq^n}{1}}}}}} = \frac{P_{n-1}(aq)}{P_n(a)},$$

where

$$P_n(a) = \sum_{k\geq 0} a^k q^{k^2} \begin{bmatrix} n+1-k \\ k \end{bmatrix}_q .$$

We can check that

$$P_o(a) = 1, \quad P_1(a) = 1 + aq,$$

and we can verify that

$$P_n(a) = P_{n-1'}(aq) + aqP_{n-2}(aq^2).$$

If we divide by $P_{n-1}(aq)$, we obtain

$$\frac{P_n(a)}{P_{n-1}(aq)} = 1 + \frac{aq}{\left(\frac{P_{n-1}(aq)}{P_{n-2}(aq^2)}\right)},$$

or,

$$\frac{P_{n-1}(aq)}{P_n(a)} = \frac{1}{1 + aq \cdot \left(\frac{P_{n-2}(aq^2)}{P_{n-1}(aq)}\right)}.$$

Iteration then gives the desired result.
For a combinatorial interpretation of $P_n(a)$, see M. D. Hirschhorn, "Partitions and Ramanujan's continued fraction", Duke Math J. 39(1973), 789–791. (My first paper.)
Luckily for me, no one saw that this result is in Ramanujan's Notebooks, nor did they realise that it appeared in a paper by P. Kesava Menon in about 1965.

In his lost notebook, Ramanujan makes the one–line remark
If

$$G(a,\lambda) = \sum_{k\geq 0} \frac{q^{(k^2+k)/2} a^k (-\lambda/a;q)_k}{(q;q)_k(-bq;q)_k}$$

then

$$\frac{G(aq,\lambda q)}{G(a,\lambda)} = \cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{1+\ddots}}}}$$

George Andrews drew my attention to this in 1978 when he arrived in Australia, told me he had proved this result "with some difficulty", and challenged me to find the convergents. He came back from a two–week trip away, to find this note on his office door.
"I have found a formula for the convergents, but this note is too small to contain it."

If

$$P_n(a,b.\lambda) = \sum_{s,t,u\geq 0} a^s b^t \lambda^u q^{(s^2+s+t^2+t)/2+u^2+st+su+tu} \times \begin{bmatrix} n+1-s-t-u \\ u \end{bmatrix}_q \times \begin{bmatrix} \lfloor (n+1)/2 \rfloor - t - u \\ s \end{bmatrix}_q \times \begin{bmatrix} \lfloor n/2 \rfloor - s - u \\ t \end{bmatrix}_q$$

where the sum is taken over all (s,t,u) with $s+t+u \leq \lfloor (n+1)/2 \rfloor$, with the special convention that

$$\begin{bmatrix} -1 \\ 0 \end{bmatrix} = 1$$

then

$$\frac{P_{2n-2}(b, aq, \lambda q)}{P_{2n-1}(a, b, \lambda)}$$

$$= \cfrac{1}{1 + \cfrac{aq + \lambda q}{1 + \cfrac{bq + \lambda q^2}{1 + \ddots \cfrac{aq^n + \lambda q^{2n-1}}{1}}}},$$

and

$$\frac{P_{2n-1}(b, aq, \lambda q)}{P_{2n}(a, b, \lambda)}$$

$$= \cfrac{1}{1 + \cfrac{aq + \lambda q}{1 + \cfrac{bq + \lambda q^2}{1 + \ddots \cfrac{bq^n + \lambda q^{2n}}{1}}}},$$

If we let $n \to \infty$ in this, we do not get Ramanujan's result, but

$$\text{the infinite continued fraction} = \frac{P(b, aq, \lambda q)}{P(a, b, \lambda)}.$$

However, it is easy to show that

$$P(b, aq, \lambda q) = P(aq, b, \lambda q)$$

and that

$$P(a, b, \lambda) = (-bq; q)_\infty G(a, q),$$

and Ramanujan's result follows.
Ramanujan gives many special cases of this remarkable alternating continued fraction, some of which were discovered by others before the LNB was discovered.
It is possible that the finite version simplifies in some of these special cases.
We are very lucky that he gave the general result, because the several parameters enable one to prove it.