The power of q

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Part IV

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Partitions with even parts distinct

Let $ped(n)$ be the number of partitions of n with even parts distinct.
For example, if $n = 5$, the partitions are
$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+1+1+1,\ 1+1+1+1+1,$
and $ped(5) = 6$.
The generating function is

$$\sum_{n\geq 0} ped(n)q^n = \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} = (-q;q)_\infty(-q^2;q^2)_\infty.$$

It is also true that

$$\begin{aligned}\sum_{n\geq 0} ped(n)q^n &= \frac{(q^2;q^2)_\infty}{(q;q)_\infty}\frac{(q^4;q^4)_\infty}{(q^2;q^2)_\infty} = \frac{(q^4;q^4)_\infty}{(q;q)_\infty} \\ &= \frac{1}{(q,q^2,q^3;q^4)_\infty}\end{aligned}$$

so

$$ped(n) = p_{\{1,2,3;4\}}(n)$$

the number of partitions of n into parts not multiples of 4. ("4–regular partitions")
In the case $n = 5$, the relevant partitions are
$5,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1$
and $p_{\{1,2,3;4\}}(5) = 6 = ped(5)$.

James Sellers and I discovered that $ped(n)$ has some nice arithmetic properties.

I shall show that $ped(n)$ is divisible by 12 for (at least) one in nine values of n.

Indeed, I will prove the Ramanujan–like identity,

$$\sum_{n\geq 0} ped(9n+7)q^n = 12 \prod_{n\geq 1} \frac{(1-q^{2n})^4(1-q^{3n})^6(1-q^{4n})}{(1-q^n)^{11}}.$$

$$\sum_{n\geq 0} ped(n)q^n = (-q;q)_\infty(-q^2;q^2)_\infty$$

$$= \frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(-q,-q^2,q^3;q^3)_\infty$$

$$\times\frac{(-q^6;q^6)_\infty}{(q^6;q^6)_\infty}(-q^2,-q^4,q^6;q^6)_\infty$$

$$= \frac{(q^6;q^6)_\infty}{(q^3;q^3)_\infty^2}\frac{(q^{12};q^{12})_\infty}{(q^6;q^6)_\infty^2}\sum_{m,n=-\infty}^{\infty} q^{(3m^2+m)/2+(3n^2+n)}.$$

Split the sum according to the residue modulo 3 of $m+2n$.

If $m+2n \equiv 0$ let $m = r-2t,\ n = r+t$,
if $m+2n \equiv 1$, let $m = r+1-2t,\ n = r+t$,
if $m+2n \equiv -1$, let $m = r-1-2t,\ n = r+t$
and we obtain

$$\sum_{n\geq 0} ped(n)q^n = \frac{(q^{12};q^{12})_\infty}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty} \times \left(\sum_{r,t=-\infty}^{\infty} q^{(9r^2+3r)/2+9t^2} + q^2 \sum_{r,t=-\infty}^{\infty} q^{(9r^2+9r)/2+(9t^2-6t)} + q \sum_{r,t=-\infty}^{\infty} q^{(9r^2-3r)/2+(9t^2+6t)} \right)$$

$$\sum_{n\geq 0} ped(n)q^n = \frac{(q^{12};q^{12})_\infty}{(q^3;q^3)_\infty^2(q^6;q^6)_\infty}$$
$$\times\left((-q^3,-q^6,q^9;q^9)_\infty(-q^9,-q^9,q^{18};q^{18})_\infty\right.$$
$$+q(-q^3,-q^6,q^9;q^9)_\infty(-q^3,-q^{15},q^{18};q^{18})_\infty$$
$$\left.+2q^2(-q^9,-q^9,q^9;q^9)_\infty(-q^3,-q^{15},q^{18};q^{18})_\infty\right)$$

$$= \frac{f_{12}}{f_3^2 f_6}\left(\frac{f_6 f_9^2}{f_3 f_{18}}\frac{f_{18}^5}{f_9^2 f_{36}^2} + q\frac{f_6 f_9^2}{f_3 f_{18}}\frac{f_6^2 f_9 f_{36}}{f_3 f_{12} f_{18}} + 2q^2\frac{f_{18}^2}{f_9}\frac{f_6^2 f_9 f_{36}}{f_2 f_{12} f_{18}}\right)$$

$$= \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q\frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2\frac{f_{18} f_{36}}{f_3^3}.$$

It follows that

$$\sum_{n\geq 0} ped(3n)q^n = \frac{f_4 f_6^4}{f_1^3 f_{12}^2},$$

$$\sum_{n\geq 0} ped(3n+1)q^n = \frac{f_2^2 f_3^3 f_{12}}{f_1^4 f_6^2},$$

$$\sum_{n\geq 0} ped(3n+2)q^n = 2\frac{f_2 f_6 f_{12}}{f_1^3}.$$

In particular,

$$\sum_{n\geq 0} ped(3n+1)q^n = \frac{f_3^3 f_{12}}{f_6^2}\cdot\frac{f_2^2}{f_1^4} = \frac{\phi(-q^3)\psi(-q^3)}{\phi(-q)^2},$$

where

$$\phi(-q) = \sum_{-\infty}^{\infty}(-1)^n q^{n^2} = (q,q,q^2;q^2)_\infty = \frac{f_1^2}{f_2},$$

$$\psi(-q) = \sum_{-\infty}^{\infty}(-1)^n q^{2n^2+n} = (q,q^3,q^4;q^4)_\infty = \frac{f_1 f_4}{f_2}.$$

So

$$\sum_{n\geq 0}(-1)^n ped(3n+1)q^n = \frac{\phi(q^3)\psi(q^3)}{\phi(q)^2}.$$

Now,

$$\frac{1}{\phi(q)} = \frac{\phi(\omega q)\phi(\omega^2 q)}{\phi(q)\phi(\omega q)\phi(\omega^2 q)}.$$

The denominator,

$$\phi(q)\phi(\omega q)\phi(\omega^2 q) = \frac{\phi(q^3)^4}{\phi(q^9)}.$$

This relies on the facts that

$$\phi(q) = \frac{E(q^2)^5}{E(q)^2 E(q^4)^2}$$

and

$$E(q)E(\omega q)E(\omega^2 q) = \frac{E(q^3)^4}{E(q^9)}.$$

(We saw a similar calculation when we proved RMBI.)

Also,

$$\phi(q) = \phi(q^9) + 2qX(q^3),$$

where

$$X(q) = (-q, -q^5, q^6; q^6)_\infty.$$

So the numerator,

$$\begin{aligned}\phi(\omega q)\phi(\omega^2 q) &= (\phi(q^9) + 2\omega qX(q^3))(\phi(q^9) + 2\omega^2 qX(q^3)) \\ &= \phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2.\end{aligned}$$

It follows that

$$\sum_{n\geq 0}(-1)^n ped(3n+1)q^n = \phi(q^3)\psi(q^3)\left(\frac{\phi(q^9)}{\phi(q^3)^4}\right)^2$$
$$\times\left(\phi(q^9)^2 - 2q\phi(q^9)X(q^3) + 4q^2X(q^3)^2\right)^2$$

It follows that

$$\sum_{n\geq 0}(-1)^n ped(9n+7)q^n = \phi(q)\psi(q)\frac{\phi(q^3)^2}{\phi(q)^8}12\phi(q^3)^2X(q)^2$$
$$= 12\frac{\phi(q^3)^4\psi(q)X(q)^2}{\phi(q)^7}$$

and

$$\sum_{n\geq 0} ped(9n+7)q^n$$
$$= 12\frac{\phi(-q^3)^4\psi(-q)X(-q)^2}{\phi(-q)^7}$$
$$= 12\left(\frac{f_2}{f_1^2}\right)^7\left(\frac{f_3^2}{f_6}\right)^4\left(\frac{f_1f_4}{f_2}\right)\left(\frac{f_1f_6^2}{f_2f_3}\right)^2$$
$$= 12\frac{f_2^3f_3^6f_4}{f_1^{11}},$$

as claimed earlier.

We can also show that for $\alpha \geq 1$, modulo 3,

$$\sum_{n\geq 0}(-1)^n ped\left(3^{2\alpha}n + \frac{3^{2\alpha}-1}{8}\right)q^n \equiv \psi(q)\phi(q^3),$$

$$\sum_{n\geq 0}(-1)^n ped\left(3^{2\alpha+1}n + \frac{3^{2\alpha+2}-1}{8}\right)q^n \equiv \phi(q)\psi(q^3).$$

and that for $\alpha \geq 1$,

$$ped\left(3^{2\alpha+1}n + \frac{17\times 3^{2\alpha}-1}{8}\right) \equiv 0 \pmod 3,$$

$$ped\left(3^{2\alpha+2}n + \frac{19\times 3^{2\alpha+1}-1}{8}\right) \equiv 0 \pmod 3.$$