The power of q

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Part V

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Partitions with odd parts distinct

Let $pod(n)$ be the number of partitions of n where the odd parts are distinct.
Then

$$\sum_{n\geq 0} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

James Sellers and I noticed, and managed to prove that

$$pod(27n+26) \equiv 0 \pmod 3,$$

and more, and that is what I am going to do here.
We have

$$\sum_{n\geq 0} pod(n)q^n = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{(q^2;q^4)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty}$$

$$= \frac{(q^2;q^4)_\infty}{(q;q)_\infty} = \frac{1}{(q,q^3,q^4;q^4)_\infty} = \frac{1}{\psi(-q)},$$

so we notice that

$$pod(n) = p_{\{1,3,4;4\}}(n),$$

the number of partitions of n into parts not congruent to 2 modulo 4.

We have

$$\sum_{n\geq 0} pod(n)q^n = \frac{1}{(q,q^3,q^4;q^4)_\infty} = \frac{1}{\psi(-q)},$$

so

$$\begin{aligned}\sum_{n\geq 0}(-1)^n pod(n)q^n &= \frac{1}{\psi(q)} \\ &= \frac{\psi(\omega q)\psi(\omega^2 q)}{\psi(q)\psi(\omega q)\psi(\omega^2 q)} \\ &= \frac{\psi(q^9)}{\psi(q^3)^4}\psi(\omega q)\psi(\omega^2 q).\end{aligned}$$

Now,

$$\begin{aligned}\psi(q) &= 1+q+q^3+q^6+q^{10}+q^{15}+q^{21}+\cdots \\ &= (1+q^3+q^6+q^{15}+q^{21}+\cdots) \\ &\qquad +q(1+q^9+q^{27}+\cdots) \\ &= P(q^3)+q\psi(q^9).\end{aligned}$$

It follows that

$$\begin{aligned}\psi(\omega q)\psi(\omega^2 q) &= \left(P(q^3)+\omega q\psi(q^9)\right)\left(P(q^3)+\omega^2 q\psi(q^9)\right) \\ &= P(q^3)^2 - qP(q^3)\psi(q^9)+\psi(q^9)^2\end{aligned}$$

and

$$\begin{aligned}&\sum_{n\geq 0}(-1)^n pod(n)q^n \\ &= \frac{\psi(q^9)}{\psi(q^3)^4}\left(P(q^3)^2 - qP(q^3)\psi(q^9)+\psi(q^9)^2\right).\end{aligned}$$

It follows that

$$\sum_{n\geq 0}(-1)^n pod(3n)q^n = \frac{\psi(q^3)}{\psi(q)^4}P(q)^2,$$

$$\sum_{n\geq 0}(-1)^n pod(3n+1)q^n = \frac{\psi(q^3)^2}{\psi(q)^4}P(q),$$

$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n = \frac{\psi(q^3)^3}{\psi(q)^4}.$$

In particular,

$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n$$

$$= \psi(q^3)^3\left(\frac{\psi(q^9)}{\psi(q^3)^4}\right)^4 \times \left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right)^4$$

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(3n+2)q^n \\
= \; & \frac{\psi(q^9)^4}{\psi(q^3)^{13}}\left(P(q^2)^8 - 4qP(q^3)^7\psi(q^9) + 10q^2P(q^3)^6\psi(q^9)^2\right. \\
& -16q^3P(q^3)^5\psi(q^9)^3 + 19q^4P(q^3)^4\psi(q^9)^4 \\
& -16q^5P(q^3)^3\psi(q^9)^5 + 10q^6P(q^3)^2\psi(q^9)^6 \\
& \left. -4q^7P(q^3)\psi(q^9)^7 + q^8\psi(q^9)^8\right).
\end{aligned}$$

Hence

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \\
= \; & \frac{\psi(q^3)^4}{\psi(q)^{13}}\left(10P(q)^6\psi(q^3)^2 - 16qP(q)^3\psi(q^3)^5 + q^2\psi(q^3)^8\right).
\end{aligned}$$

Now,

$$\begin{aligned}\frac{\psi(q^3)^4}{\psi(q^9)} &= \psi(q)\psi(\omega q)\psi(\omega^2 q)\\ &= \left(P(q^3)+q\psi(q^9)\right)\left(P(q^3)+\omega q\psi(q^9)\right)\\ &\qquad\times\left(P(q^3)+\omega^2 q\psi(q^9)\right)\\ &= P(q^3)^3+q^3\psi(q^9)^3,\end{aligned}$$

from which it follows that

$$P(q)^3=\frac{\psi(q)^4-q\psi(q^3)^4}{\psi(q^3)}.$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \\
&= \frac{\psi(q^3)^4}{\psi(q)^{13}}\Bigg(10\psi(q^3)^2\left(\frac{\psi(q)^4-q\psi(q^3)^4}{\psi(q^3)}\right)^2 \\
&\qquad -16q\psi(q^3)^5\left(\frac{\psi(q)^4-q\psi(q^3)^4}{\psi(q^3)}\right) \\
&\qquad +q^2\psi(q^3)^8\Bigg) \\
&= 10\frac{\psi(q^3)^4}{\psi(q)^5}-36q\frac{\psi(q^3)^8}{\psi(q)^9}+27q^2\frac{\psi(q^3)^{12}}{\psi(q)^{13}} \\
&\equiv \frac{\psi(q^3)^4}{\psi(q)^5} \pmod 9.
\end{aligned}$$

We have (modulo 9)

$$
\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \\
\equiv\ & \psi(q^3)^4\left(\frac{\psi(q^9)}{\psi(q^3)^4}\right)^5 \\
&\quad \times\left(P(q^3)^2 - qP(q^3)\psi(q^9) + q^2\psi(q^9)^2\right)^5 \\
\equiv\ & \frac{\psi(q^9)^5}{\psi(q^3)^{16}}\Big(P(q^3)^{10} - 5qP(q^3)^9\psi(q^9) \\
&\quad +15q^2P(q^3)^8\psi(q^9)^2 - 30q^3P(q^3)^7\psi(q^9)^3 \\
&\quad +45q^4P(q^3)^6\psi(q^9)^4 - 51q^5P(q^3)^5\psi(q^9)^5 \\
&\quad +45q^6P(q^3)^4\psi(q^9)^6 - 30q^7P(q^3)^3\psi(q^9)^7 \\
&\quad +15q^8P(q^3)^2\psi(q^9)^8 - 5q^9P(q^3)\psi(q^9)^9 \\
&\quad +q^{10}\psi(q^9)^{10}\Big).
\end{aligned}
$$

It follows that, modulo 9,

$$\sum_{n\geq 0}(-1)^n pod(27n+26)q^n$$
$$\equiv \frac{\psi(q^3)^5}{\psi(q)^{16}}\Big(15P(q)^8\psi(q^3)^8 - 51qP(q)^5\psi(q^3)^5 + 15q^2P(q)^2\psi(q^3)^8\Big),$$

It follows that

$$\sum_{n\geq 0}(-1)^n pod(27n+26)q^n \equiv 0 \pmod 3,$$

and so

$$pod(27n+26) \equiv 0 \pmod 3,$$

which was my first goal.

We also find that, modulo 9,

$$\begin{aligned}
&\sum_{n\geq 0}(-1)^n pod(27n+17)q^n \\
\equiv\ & \frac{\psi(q^3)^5}{\psi(q)^{16}}\Big(5P(q)^9\psi(q^3) - 45qP(q)^6\psi(q^3)^4 \\
&\qquad +30q^2P(q)^3\psi(q^3)^7 - q^3\psi(q^3)^{10}\Big) \\
=\ & \frac{\psi(q^3)^5}{\psi(q)^{16}}\Bigg(5\psi(q^3)\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right)^3 \\
&\qquad -45q\psi(q^3)^4\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right)^2 \\
&\qquad +30q^2\psi(q^3)^7\left(\frac{\psi(q)^4 - q\psi(q^3)^4}{\psi(q^3)}\right) \\
&\qquad -q^3\psi(q^3)^{10}\Bigg)
\end{aligned}$$

That is, modulo 9,

$$\sum_{n\geq 0}(-1)^n pod(27n+17)q^n$$
$$\equiv \quad 5\frac{\psi(q^3)^3}{\psi(q)^4} - 60q\frac{\psi(q^3)^7}{\psi(q)^8} + 135q^2\frac{\psi(q^3)^{11}}{\psi(q)^{12}} - 81q^3\frac{\psi(q^3)^{15}}{\psi(q)^{16}}$$

and so, modulo 3 (not 9, though),

$$\sum_{n\geq 0}(-1)^n pod(27n+17)q^n \equiv -\frac{\psi(q^3)^3}{\psi(q)^4}.$$

We can go on to prove by induction on α that for $\alpha \geq 0$,

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+1}n+\frac{5\times 3^{2\alpha+1}+1}{8}\right)q^n \equiv (-1)^\alpha \frac{\psi(q^3)^3}{\psi(q)^4} \pmod 3,$$

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n+\frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n \equiv (-1)^\alpha \frac{\psi(q^3)^4}{\psi(q)^5} \pmod 3,$$

and that for $\alpha \geq 0$,

$$pod\left(3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right) \equiv 0 \pmod 3.$$

Florian Luca pointed out that all the above can be shortened, as follows.
We have, modulo 3,

$$\sum_{n\geq 0}(-1)^n pod(n)q^n = \frac{1}{\psi(q)} \equiv \frac{\psi(q)^2}{\psi(q^3)}$$
$$\equiv \frac{\left(P(q^3)+q\psi(q^9)\right)^2}{\psi(q^3)}$$

It follows that

$$\sum_{n\geq 0}(-1)^n pod(3n)q^n \equiv \frac{P(q)^2}{\psi(q)},$$
$$\sum_{n\geq 0}(-1)^n pod(3n+1)q^n \equiv \frac{\psi(q^3)P(q)}{\psi(q)},$$
$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n \equiv \frac{\psi(q^3)^2}{\psi(q)}.$$

In particular,

$$\sum_{n\geq 0}(-1)^n pod(3n+2)q^n \equiv \frac{\psi(q^3)^2}{\psi(q)} \equiv \psi(q^3)\psi(q)^2$$
$$\equiv \psi(q^3)\left(P(q^3)+q\psi(q^9)\right)^2$$

so

$$\sum_{n\geq 0}(-1)^n pod(9n+8)q^n \equiv \psi(q^3)^2\psi(q)$$
$$\equiv \psi(q^3)^2\left(P(q^3)+q\psi(q^9)\right)$$

so

$$\sum_{n\geq 0}(-1)^n pod(27n+17)q^n \equiv 0.$$

Further, we can prove by induction that for $\alpha \geq 0$,

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+1}n+\frac{5\times 3^{2\alpha+1}+1}{8}\right)q^n \equiv (-1)^\alpha \psi(q^3)\psi(q)^2 \pmod 3,$$

$$\sum_{n\geq 0}(-1)^n pod\left(3^{2\alpha+2}n+\frac{7\times 3^{2\alpha+2}+1}{8}\right)q^n \equiv (-1)^\alpha \psi(q^3)^2\psi(q) \pmod 3,$$

and that for $\alpha \geq 0$,

$$pod\left(3^{2\alpha+3}n+\frac{23\times 3^{2\alpha+2}+1}{8}\right)\equiv 0 \pmod 3.$$