The power of q

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A course of lectures presented at Wits, July 2014
Part VI

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The case of the mysterious sevens

Consider the following table

n	number of partitions into distinct odd parts	number of partitions into distinct even parts
1	1	0
2	0	1
3	1	0
4	1	1
5	1	0
6	1	2
7	1	0
8	2	2
9	2	0
10	2	3
11	2	0
12	3	4
13	3	0
14	4	5

I want to draw your attention to the following sub-table.

$2n+1$	into distinct odd parts	$2n$	into distinct even parts
3	1	2	1
5	1	4	1
7	1	6	2
9	2	8	2
11	2	10	3
13	3	12	4

Notice that the number of partitions of $2n+1$ into distinct odd parts seems always to be equal to, or one less than, the number of partitions of $2n$ into distinct even parts.

Let us examine the cases where the first number is one less than the second.

$$7 := 7 \quad \text{(one partition)}$$

$$11 := 11 = 7+3+1 \quad \text{(2 partitions)}$$

$$13 := 13 = 9+3+1 = 7+5+1 \quad \text{(3 partitions)}$$

Now notice that if we allow 7's of a different colour, **7**, then in each case the number of partitions is boosted by 1, and is now equal to the number of partitions of the number one smaller into distinct even parts.

$$7 := 7 = \mathbf{7}, \quad 6 := 6 = 4 + 2$$

$$11 := 11 = 7+3+1 = \mathbf{7}+3+1, \quad 10 := 10 = 8+2 = 6+4$$

$$13 := 13 = 9+3+1 = 7+5+1 = \mathbf{7}+5+1,$$
$$12 := 12 = 10+2 = 8+4 = 6+4+2.$$

Maybe that always works?

Let us continue the table, allowing 7's of two colours.

	distinct odd parts with 7's of two colours		distinct even parts
$2n+1$		$2n$	
13	4	12	4
15	6	14	5
17	7	16	6

Well, that didn't work! Let's check.

$$15 := 15 = 11+3+1 = 9+5+1 = 7+\mathbf{7}+1$$
$$= 7+5+3 = \mathbf{7}+5+3$$
$$14 := 14 = 12+2 = 10+4 = 8+6 = 8+4+2,$$

$$17 := 17 = 13+3+1 = 11+5+1 = 9+7+1$$
$$= 9+\mathbf{7}+1 = 9+5+3 = 7+\mathbf{7}+3,$$
$$16 := 16 = 14+2 = 12+4 = 10+6 = 10+4+2$$
$$= 8+6+2.$$

Now the balance has swung the other way!
How can we compensate?

To cut a long story short, let us throw into the mix all multiples of 7 in the second colour.
The conjecture is,
if S is the set of all numbers in one colour together with multiples of 7 in another colour, then the number of partitions of $2n+1$ into distinct odd elements of S is equal to the number of partitions of $2n$ into distinct even elements of S.

And, indeed, we shall show that this is a Theorem!
We have to show that

$$\mathbf{O}\left(\prod_{n\geq 1}(1+q^{2n-1})(1+q^{14n-7})\right) = q\prod_{n\geq 1}(1+q^{2n})(1+q^{14n}).$$

That is,

$$\mathbf{O}\left((-q;q^2)_\infty(-q^7;q^{14})_\infty\right) = q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty.$$

where $\mathbf{O}$ denotes "**O**dd part".

Now,

$$\begin{aligned}(-q;q^2)_\infty &= \frac{1}{(q^4;q^4)_\infty}(-q,-q^3,q^4;q^4)_\infty \\ &= \frac{1}{(q^4;q^4)_\infty}\sum_{-\infty}^{\infty} q^{2n^2+n} \\ &= \frac{1}{(q^4;q^4)_\infty}\left(\sum_{-\infty}^{\infty} q^{8n^2+2n} + q\sum_{-\infty}^{\infty} q^{8n^2+6n}\right)\end{aligned}$$

Similarly,

$$(-q^7;q^{14})_\infty = \frac{1}{(q^{28};q^{28})_\infty}\left(q^{56n^2+14n} + q^7\sum_{-\infty}^{\infty} q^{56n^2+42n}\right).$$

If we multiply these together and extract the odd part, we find

$$\begin{aligned}
&\mathbf{O}\left((-q;q^2)_\infty(-q^7;q^{14})_\infty\right) \\
= \quad &\frac{1}{(q^4;q^4)_\infty(q^{28};q^{28})_\infty} \\
&\times\left(q\sum_{m,n=-\infty}^{\infty} q^{8m^2+6m+56n^2+14n}\right. \\
&\qquad\qquad \left.+q^7\sum_{-\infty}^{\infty} q^{8m^2+2m+56n^2+42n}\right).
\end{aligned}$$

Now let us examine the right side.
We have

$$\begin{aligned}(-q^2;q^2)_\infty &= (-q^2,-q^4,-q^6,-q^8;q^8)_\infty \\ &= (-q^4;q^4)_\infty(-q^2,-q^6;q^8)_\infty \\ &= \frac{(-q^4;q^4)_\infty}{(q^8;q^8)_\infty}(-q^2,-q^6,q^8;q^8)_\infty \\ &= \frac{1}{(q^4;q^4)_\infty}\sum_{-\infty}^{\infty} q^{4n^2+2n}.\end{aligned}$$

So the right side is

$$\begin{aligned}& q(-q^2;q^2)_\infty(-q^{14};q^{14})_\infty \\ = \; & \frac{1}{(q^4;q^4)_\infty(q^{28};q^{28})_\infty} \\ & \times q \sum_{m,n=-\infty}^{\infty} q^{4m^2+2m+28n^2+14n}.\end{aligned}$$

So, we want to show that

$$\sum_{-\infty}^{\infty} q^{8m^2+6m+56n^2+14n} + q^6 \sum_{-\infty}^{\infty} q^{8m^2+2m+56n^2+42n}$$
$$= \sum_{-\infty}^{\infty} q^{4m^2+2m+28n^2+14n}.$$

Replace q by q^8 and multiply by q^{16}, and this becomes

$$\sum_{-\infty}^{\infty} q^{(8m-3)^2+7(8n+1)^2} + \sum_{-\infty}^{\infty} q^{(8m+1)^2+7(8n-3)^2}$$
$$= \sum_{-\infty}^{\infty} q^{2\left((4m+1)^2+7(4n+1)^2\right)}.$$

Note that I have cunningly used the fact that

$$\sum_{-\infty}^{\infty} q^{(8m+3)^2} = \sum_{-\infty}^{\infty} q^{(8m-3)^2}.$$

The left side can be written

$$\sum_{-\infty}^{\infty} q^{(8m-3)^2+7(8n+1)^2} + \sum_{-\infty}^{\infty} q^{(8m+1)^2+7(8n-3)^2}$$

$$= \sum_{u,v=-\infty}^{\infty} q^{(4u+1)^2+7(4v+1)^2}$$

with the extra condition that $u - v$ is odd.

So we want to show that

$$\sum_{\substack{u,v=-\infty \\ u-v\equiv 1 \pmod 2}}^{\infty} q^{(4u+1)^2+7(4v+1)^2} = \sum_{-\infty}^{\infty} q^{2\left((4m+1)^2+7(4n+1)^2\right)}.$$

We split the sum on the left according to the residue of $u - v$ modulo 8.

If $u - v \equiv -1 \pmod 8$, write $u = k - 7l - 1, \quad v = k + l$, then the index,

$$\begin{aligned}(4u+1)^2 + 7(4v+1)^2 &= (4k - 28l - 3)^2 + 7(4k + 4l + 1)^2 \\ &= 128k^2 + 896l^2 + 32k + 224l + 16 \\ &= 2\left((8k+1)^2 + 7(8l+1)^2\right) \\ &= 2\left((4m+1)^2 + 7(4n+1)^2\right)\end{aligned}$$

with $m, \ n$ **both even.**

If $u - v \equiv 3 \pmod 8$, write $u = k - 7l + 3, \quad v = k + l$, and the index

$$\begin{aligned}(4u+1)^2 + 7(4v+1)^2 &= 2\left((8k+5)^2 + 7(8l-3)^2\right) \\ &= 2\left((4m+1)^2 + 7(4n+1)^2\right)\end{aligned}$$

with $m, \ n$ **both odd.**

If $u-v\equiv 1 \pmod 8$, write $u=-k+7l+1,\ v=-k-l$, and the index is

$$\begin{aligned}(4u+1)^2+7(4v+1)^2 &= 2\left((8k-3)^2+7(8l+1)^2\right)\\ &= 2\left((4m+1)^2+7(4n+1)^2\right)\end{aligned}$$

with m odd, n even.

Finally, if $u-v\equiv -3 \pmod 8$, write
$u=-k+7l-3,\ v=-k-l$, and the index

$$\begin{aligned}(4u+1)^2+7(4v+1)^2 &= 2\left(8k+1)^2+7(8l-3)^2\right)\\ &= 2\left((4m+1)^2+7(4n+1)^2\right)\end{aligned}$$

with m even, n odd.

And the theorem is proved!