A letter from Fitzroy House

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Introduction

On a Saturday in 1917 or 1918, Ramanujan sent Hardy a letter from the nursing home Fitzroy House, in which he stated a particular identity. He hoped it might be helpful in solving Waring's Problem for fourth powers. (It has, it appears, never been taken up for this purpose.)

Of course, Ramanujan did not give any indication of a proof.

The object of my talk is to give a proof, from scratch.

The details will get a little ferocious, but the ideas are simple. You will need little other than induction and the product rule of differentiation to follow the proof.

Jacobi's Triple Product Identity

Before stating this identity explicitly, let us begin by trying to expand the product

$$\begin{aligned} F(a) &= \prod_{1}^{\infty}(1+a^{-1}q^{2n-1})(1+aq^{2n-1}) \\ &= (1+a^{-1}q)(1+aq)(1+a^{-1}q^3)(1+aq^3)\ \cdots\ . \end{aligned}$$

Observe that

$$F(aq^2) = (1+a^{-1}q^{-1})(1+aq^3)(1+a^{-1}q)(1+aq^5)\ \cdots\ .$$

You see that

$$F(a) = \frac{1+aq}{1+a^{-1}q^{-1}}F(aq^2).$$

That is,

$$F(a) = aqF(aq^2).$$

Now suppose that

$$F(a) = \sum_{-\infty}^{\infty} c_n(q)a^n.$$

By symmetry $(F(a^{-1}) = F(a))$,

$$c_{-n} = c_n.$$

Also,

$$\sum_{-\infty}^{\infty} c_n(q)a^n = aq\sum_{-\infty}^{\infty} c_n(q)q^{2n}a^n = \sum_{-\infty}^{\infty} q^{2n-1}c_{n-1}a^n.$$

It follows (by comparing coefficients of a^n) that for all n,

$$c_n = q^{2n-1}c_{n-1}.$$

It follows by induction that for all n,

$$c_n = q^{n^2} c_0.$$

So

$$F(a) = c_0 \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

That is,

$$\prod_1^{\infty} (1 + a^{-1} q^{2n-1})(1 + aq^{2n-1}) = c_0(q) \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

Our problem now is to determine $c_0(q)$.

We have

$$\prod_1^\infty (1+a^{-1}q^{2n-1})(1+aq^{2n-1}) = c_0(q)\sum_{-\infty}^{\infty} a^n q^{n^2}.$$

With q replaced by q^4 and a by q^2, this gives

$$\prod_1^\infty (1+q^{4n-2}) = c_0(q^4)\sum_{-\infty}^{\infty} q^{4n^2+2n},$$

while if instead we simply replace a by q and divide by two, we obtain

$$\prod_1^\infty (1+q^{2n})^2 = c_0(q)\sum_{0}^{\infty} q^{n^2+n}.$$

But, surprisingly perhaps,

$$\sum_{0}^{\infty} q^{n^2+n} = \sum_{-\infty}^{\infty} q^{4n^2+2n}.$$

We have

$$c_0(q)\sum_0^\infty q^{n^2+n} = \prod_1^\infty (1+q^{2n})^2$$

and

$$c_0(q^4)\sum_{-\infty}^\infty q^{4n^2+2n} = \prod_1^\infty (1+q^{4n-2}).$$

It follows by division that

$$\frac{c_0(q)}{c_0(q^4)} = \prod_1^\infty \frac{(1+q^{2n})^2}{(1+q^{4n-2})} = \prod_1^\infty \frac{(1+q^{2n})^2(1+q^{4n})}{(1+q^{4n-2})(1+q^{4n})}$$

$$= \prod_1^\infty (1+q^{2n})(1+q^{4n}) = \prod_1^\infty \frac{(1-q^{4n})(1-q^{8n})}{(1-q^{2n})(1-q^{4n})}$$

$$= \prod_1^\infty \frac{1-q^{8n}}{1-q^{2n}}.$$

So

$$\prod_1^\infty (1-q^{2n})c_0(q) = \prod_1^\infty (1-q^{8n})c_0(q^4).$$

It follows by induction on k that

$$\prod_1^\infty (1-q^{2n})c_0(q) = \prod_1^\infty (1-q^{2\times 4^k n})c_0(q^{4^k}).$$

If we now let $k \to \infty$, the left side is constant, while the right side $\to c_0(0) = 1$,
so

$$\prod_1^\infty (1-q^{2n})c_0(q) = 1,$$

$$c_0(q) = 1/\prod_1^\infty (1-q^{2n}),$$

$$\prod_1^\infty (1+a^{-1}q^{2n-1})(1+aq^{2n-1}) = \sum_{-\infty}^{\infty} a^n q^{n^2} / \prod_1^\infty (1-q^{2n})$$

and we obtain the justly celebrated Triple Product Identity of Jacobi,

$$\prod_{1}^{\infty}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n}) = \sum_{-\infty}^{\infty} a^n q^{n^2}.$$

Ramanujan now considers the sum

$$s(q) = \sum_{k,l=-\infty}^{\infty} q^{k^2+kl+l^2}.$$

The identity in the letter

The claim Ramanujan makes in the letter to Hardy is that

$$\sum_{k,l=-\infty}^{\infty} q^{k^2+kl+l^2} = 1 + 6\sum_{1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}} - \frac{q^{3n-1}}{1-q^{3n-1}}\right).$$

This is reminiscent of the two–squares identity,

$$\sum_{k,l=-\infty}^{\infty} q^{k^2+l^2} = 1 + 4\sum_{1}^{\infty}\left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}}\right).$$

Let us begin by considering

$$\begin{aligned} s(q^2) &= \sum_{k,l=-\infty}^{\infty} q^{2k^2+2kl+2l^2} = \sum_{k.l=-\infty}^{\infty} q^{k^2+l^2+(-k-l)^2} \\ &= \sum_{r+s+t=0} q^{r^2+s^2+t^2}. \end{aligned}$$

Consider the cube of Jacobi's triple product identity,

$$G(a) = \left(\prod_{1}^{\infty}(1+a^{-1}q^{2n-1})(1+aq^{2n-1})(1-q^{2n})\right)^3$$
$$= \left(\sum_{-\infty}^{\infty} a^n q^{n^2}\right)^3 = \sum_{r,s,t=-\infty}^{\infty} a^{r+s+t} q^{r^2+s^2+t^2}.$$

We see that $s(q^2)$ is the Constant Term with respect to a of $G(a)$.

So we want to find an alternative, possibly simpler, form for this constant term.

We start by observing that

$$G(a) = \left(\frac{1+aq}{1+a^{-1}q^{-1}}\right)^3 G(aq^2) = a^3 q^3 G(aq^2).$$

Suppose

$$G(a) = \sum_{-\infty}^{\infty} d_n(q) a^n.$$

Note that, by symmetry,

$$d_{-n} = d_n.$$

Also,

$$d_n = q^{2n-3} d_{n-3}.$$

It follows by induction that for all n,

$$d_{3n} = q^{3n^2} d_0, \quad d_{3n+1} = q^{3n^2+2n} d_1,$$
$$d_{3n-1} = q^{3n^2-2n} d_{-1} = q^{3n^2-2n} d_1,$$

$$\begin{aligned} G(a) &= d_0 \sum_{-\infty}^{\infty} q^{3n^2} \\ &\quad + d_1 \left(\sum_{-\infty}^{\infty} a^{3n+1} q^{3n^2+2n} + \sum_{-\infty}^{\infty} a^{3n-1} q^{3n^2-2n} \right). \end{aligned}$$

We make use of Jacobi's triple product to get

$$
\begin{aligned}
&G(a)\\
&=\prod_{1}^{\infty}(1+a^{-1}q^{2n-1})^3(1+aq^{2n-1})^3(1-q^{2n})^3\\
&=d_0\prod_{1}^{\infty}(1+a^{-3}q^{6n-3})(1+a^3q^{6n-3})(1-q^{6n})\\
&+d_1\left(a\prod_{1}^{\infty}(1+a^{-3}q^{6n-5})(1+a^3q^{6n-1})(1-q^{6n})\right.\\
&\qquad\left.+a^{-1}\prod_{1}^{\infty}(1+a^{-3}q^{6n-1})(1+a^3q^{6n-5})(1-q^{6n})\right).
\end{aligned}
$$

We now have the problem of determining d_0 and d_1.

We replace a by a^2q and multiply by a^3 to get

$$(a+a^{-1})^3\prod_1^\infty(1+a^{-2}q^{2n})^3(1+a^2q^{2n})^3(1-q^{2n})^3$$

$$=d_0(a^3+a^{-3})\prod_1^\infty(1+a^{-6}q^{6n})(1+a^6q^{6n})(1-q^{6n})$$

$$+d_1q^{-1}\left(a\prod_1^\infty(1+a^{-6}q^{6n-4})(1+a^6q^{6n-2})(1-q^{6n})\right.$$

$$\left.+a^{-1}\prod_1^\infty(1+a^{-6}q^{6n-2})(1+a^6q^{6n-4})(1-q^{6n})\right)$$

If we set $a=\exp\left\{\frac{\pi i}{6}\right\}$ then $a^3=i$, the first term on the right vanishes, and we find

$$d_1=3q\prod_1^\infty\frac{(1-q^{6n})^3}{(1-q^{2n})}.$$

So we have

$$(a+a^{-1})^3\prod_1^\infty(1+a^{-2}q^{2n})^3(1+a^2q^{2n})^3(1-q^{2n})^3$$

$$=d_0(a^3+a^{-3})\prod_1^\infty(1+a^{-6}q^{6n})(1+a^6q^{6n})(1-q^{6n})$$

$$+3\prod_1^\infty\frac{(1-q^{6n})^3}{(1-q^{2n})}\left(a\prod_1^\infty(1+a^{-6}q^{6n})(1+a^6q^{6n})(1-q^{6n})\right.$$

$$\left.+a^{-1}\prod_1^\infty(1+a^{-6}q^{6n-2})(1+a^6q^{6n-4})(1-q^{6n})\right).$$

We are now going to differentiate with respect to a, multiply by a, then set $a=i$.

We use the product rule, which can be stated as follows.

$$\text{If } P=\prod_n u_n \text{ then } P'=P\sum_n\frac{u_n'}{u_n}.$$

When we differentiate the left side w. r. to a and multiply by a, it becomes

$$\prod_1^\infty (1+a^{-2}q^{2n})^3(1+a^2q^{2n})^3(1-q^{2n})^3$$
$$\times\Big(3(a+a^{-1})^2(a-a^{-1})$$
$$+(a+a^{-1})^3\sum_1^\infty\left(\frac{6a^2q^{2n}}{1+a^2q^{2n}}-\frac{6a^{-2}q^{2n}}{1+a^{-2}q^{2n}}\right)\Big).$$

When we set $a=i$, this becomes zero.

When we differentiate the right side w. r. to a and multiply by a we get

$$d_0 \prod_1^\infty (1+a^{-6}q^{6n})(1+a^6q^{6n})(1-q^{6n})\Big(3(a^3-a^{-3})$$

$$+(a^3+a^{-3})\sum_1^\infty \left(\frac{6a^6q^{6n}}{1+a^6q^{6n}} - \frac{6a^{-6}q^{6n}}{1+a^{-6}q^{6n})}\right)\Big)$$

$$+3a\prod_1^\infty \frac{(1-q^{6n})^4}{(1-q^{2n})}(1+a^{-6}q^{6n-4})(1+a^6q^{6n-2})$$

$$\times\left(1+\sum_1^\infty\left(\frac{6a^6q^{6n-2}}{1+a^6q^{6n-2}} - \frac{6a^{-6}q^{6n-4}}{1+a^{-6}q^{6n-4}}\right)\right)$$

$$-3a^{-1}\prod_1^\infty \frac{(1-q^{6n})^4}{(1-q^{2n})}(1+a^{-6}q^{6n-2})(1+a^6q^{6n-4})$$

$$\times\left(1-\sum_1^\infty\left(\frac{6a^6q^{6n-4}}{1+a^6q^{6n-4}} - \frac{6a^{-6}q^{6n-2}}{1+a^{-6}q^{6n-2}}\right)\right).$$

If in this we set $a = i$ and set the result equal to zero, we find

$$d_0 = 1 + 6 \sum_{1}^{\infty} \left(\frac{q^{6n-4}}{1 - q^{6n-4}} - \frac{q^{6n-2}}{1 - q^{6n-2}} \right),$$

and

$$s(q^2) = \mathrm{CT}_a(G(a)) = 1 + 6 \sum_{1}^{\infty} \left(\frac{q^{6n-4}}{1 - q^{6n-4}} - \frac{q^{6n-2}}{1 - q^{6n-2}} \right).$$

If we now replace q^2 by q, we obtain

$$s(q) = 1 + 6 \sum_{1}^{\infty} \left(\frac{q^{3n-2}}{1 - q^{3n-2}} - \frac{q^{3n-1}}{1 - q^{3n-1}} \right),$$

which is Ramanujan's result!

Ramanujan's result tells us that if we write

$$\sum_{k,l=-\infty}^{\infty} q^{k^2+kl+l^2}$$

as a series,
the coefficient of q^n is 6 times the difference between the number of divisors of n which are congruent to $+1$ modulo 3 and the number of divisors of n which are congruent to -1 modulo 3.

Thus, for example if $n = 7$, there are two divisors of n that are congruent to $+1$ modulo 3 (1 and 7), and none that are congruent to -1 modulo 3, so 7 has $6 \times (2 - 0) = 12$ representations in the form $k^2 + kl + l^2$. Indeed, they are given by

$$(k,l) = (1,2),\ (2,1),\ (-1,-2),\ (-2,-1),\ (3,-1),\ (3,-2),$$
$$(-1,3),\ (-2,3),\ (-3,1),\ (-3,2), (1,-3), (2,-3).$$

Thank you for your attendance and your attention!