Mike's Maths Mysteries Problem 1

Solution

Suppose the radii of the two larger circles are L and R.

Suppose the line is the axis, that the centre of the circle on the left is $(0, L)$, the centre of the right circle is (X, R). Then X is determined by

$$X^2 + (L - R)^2 = (L + R)^2,$$

so

$$X = 2\sqrt{LR}.$$

Now suppose the centre of the little circle is (x, r). Then we have both

$$x^2 + (L - r)^2 = (L + r)^2 \quad \text{and} \quad (X - x)^2 + (R - r)^2 = (R + r)^2.$$

That is,

$$x^2 = 4Lr \quad \text{and} \quad x^2 - 2Xx = 4Rr - X^2.$$

By subtracting the second of these equation from the first, we find

$$2Xx = 4(L - R)r + X^2,$$

$$x = \frac{4(L - R)r + X^2}{2X},$$

and so

$$\left(\frac{4(L - R)r + X^2}{2X}\right)^2 = 4Lr.$$

That is,

$$(4(L - R)r + 4LR)^2 = 16X^2Lr = 64L^2Rr,$$

$$((L - R)r + LR)^2 = 4L^2Rr,$$

$$(L - R)^2r^2 + 2LR(L - R)r + L^2R^2 = 4L^2Rr,$$

and

$$(L - R)^2r^2 - 2LR(L + R)r + L^2R^2 = 0.$$

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To solve the quadratic, we cunningly write

$$L^2R^2\left(\frac{1}{r}\right)^2 - 2LR(L+R)\left(\frac{1}{r}\right) + (L-R)^2 = 0,$$

$$\left(\frac{LR}{r} - (L+R)\right)^2 = (L+R)^2 - (L-R)^2 = 4LR,$$

$$\frac{LR}{r} - (L+R) = \pm 2\sqrt{LR},$$

$$\frac{LR}{r} = (L+R) \pm 2\sqrt{LR},$$

$$\frac{1}{r} = \frac{1}{L} + \frac{1}{R} \pm \frac{2}{\sqrt{LR}} = \left(\frac{1}{\sqrt{L}} \pm \frac{1}{\sqrt{R}}\right)^2,$$

$$\frac{1}{\sqrt{r}} = \left|\frac{1}{\sqrt{L}} \pm \frac{1}{\sqrt{R}}\right|,$$

and since r is symmetric in L and R, and smaller than both,

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{L}} + \frac{1}{\sqrt{R}}.$$