

Mike's Maths Mysteries Problem 2

Solution

Let the radii of the small circles be r_1, r_2, \dots . Then, by the solution to MMM1,

$$\frac{1}{\sqrt{r_1}} = 2,$$

and for $n \geq 1$,

$$\frac{1}{\sqrt{r_{n+1}}} = \frac{1}{\sqrt{r_n}} + 1.$$

It follows by induction that

$$\frac{1}{\sqrt{r_n}} = n + 1$$

and

$$r_n = \frac{1}{(n+1)^2}.$$

It follows that the sum of the areas of the small circles is

$$\begin{aligned} \pi(r_1^2 + r_2^2 + \dots) &= \pi \left(\frac{1}{2^4} + \frac{1}{3^4} + \dots \right) \\ &= \pi \sum_{n=2}^{\infty} \frac{1}{n^4} \\ &= \pi \left(\frac{\pi^4}{90} - 1 \right) \\ &= \frac{\pi}{90} (\pi^4 - 90). \end{aligned}$$