Mike's Maths Mysteries Problem 3

Solution

First of all, the equation $z^8 = 16$ has the eight solutions

$$z = \sqrt{2}, \quad 1+i, \quad \sqrt{2}i, \quad -1+i, \quad -\sqrt{2}, \quad -1-i, \quad -\sqrt{2}i, \quad 1-i,$$

so

$$\begin{aligned} z^8 - 16 &= (z-\sqrt{2})(z+\sqrt{2})(z-\sqrt{2}i)(z+\sqrt{2}i)(z-(1+i))(z-(1-i))(z-(-1+i))(z-(-1-i)) \\ &= (z-\sqrt{2})(z+\sqrt{2})(z^2+2)(z^2-2z+2)(z^2+2z+2). \end{aligned}$$

Therefore

$$I_k = \int_0^1 \frac{x^k}{16-x^8}\,dx = \int_0^1 \frac{x^k}{(\sqrt{2}-x)(\sqrt{2}+x)(x^2+2)(x^2-2x+2)(x^2+2x+2)}\,dx.$$

So, using partial fractions,

$$\begin{aligned} I_0 &= \int_0^1 \frac{1}{(\sqrt{2}-x)(\sqrt{2}+x)(x^2+2)(x^2-2x+2)(x^2+2x+2)}\,dx \\ &= \int_0^1 \frac{\sqrt{2}}{128}\frac{1}{\sqrt{2}-x} + \frac{\sqrt{2}}{128}\frac{1}{\sqrt{2}+x} + \frac{1}{32}\frac{1}{x^2+2} + \frac{1}{64}\frac{2-x}{x^2-2x+2} + \frac{1}{64}\frac{1}{x^2+2x+2}\,dx \\ &= \Big[-\frac{\sqrt{2}}{128}\log(2-x) - \frac{\sqrt{2}}{128}\log(2+x) + \frac{\sqrt{2}}{64}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{1}{128}\log(x^2-2x+2) \\ &\qquad - \frac{1}{128}\log(x^2-2x+2) + \frac{1}{64}\tan^{-1}(x-1) + \frac{1}{64}\tan^{-1}(x-1)\Big]_0^1 \\ &= \frac{1}{128}\log 5 + \frac{\sqrt{2}}{64}\log(\sqrt{2}+1) + \frac{\sqrt{2}}{64}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{1}{64}\tan^{-1} 2, \end{aligned}$$

Similarly,

$$I_1 = \frac{1}{64}\pi + \frac{1}{64}\log 3 - \frac{1}{32}\tan^{-1} 2,$$

$$I_2 = -\frac{1}{64}\log 5 - \frac{\sqrt{2}}{32}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{32}\log(\sqrt{2}+1),$$

$$I_3 = -\frac{1}{32}\log 3 + \frac{1}{32}\log 5, ,$$

$$I_4 = -\frac{1}{32}\log 5 + \frac{\sqrt{2}}{16}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{16}\log(\sqrt{2}+1) - \frac{1}{16}\tan^{-1} 2,$$

$$I_5 = -\frac{1}{16}\pi + \frac{1}{16}\log 3 + \frac{1}{8}\tan^{-1} 2,$$

$$I_6 = \frac{1}{16}\log 5 - \frac{\sqrt{2}}{8}\tan^{-1}\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{8}\log(\sqrt{2}+21) - \frac{1}{8}\tan^{-1} 2,$$

$$I_7 = \frac{1}{2}\log 2 - \frac{1}{8}\log 3 - \frac{1}{8}\log 5.$$

It follows that

$$\int_0^1 \frac{16 - 8x^3 - 4x^4 - 4x^5}{16 - x^8}\, dx = 16I_0 - 8I_3 - 4I_4 - 4I_5 = \frac{\pi}{4}.$$

Therefore

$$\begin{aligned}
\frac{\pi}{4} &= \int_0^1 \frac{16 - 8x^3 - 4x^4 - 4x^5}{16 - x^2}\, dx \\
&= \int_0^1 (16 - 8x^3 - 4x^4 - 4x^5) \sum_{n\geq 0} \frac{x^{8n}}{16^{n+1}} \\
&= \sum_{n\geq 0} \frac{1}{16^{n+1}} \left(\frac{16}{8n+1} - \frac{8}{8n+4} - \frac{4}{8n+5} - \frac{4}{8n+6} \right).
\end{aligned}$$