Mike's Maths Mysteries Problem 4

Solution (third version)

Let the distance from the foot of the west wall to the point on the ground underneath the point of intersection be a, and the distance from there to the east wall be b. Then

$$a + b = w.$$

Also, from similar triangles,

$$\frac{h}{a} = \frac{\sqrt{L^2 - w^2}}{w} \quad \text{and} \quad \frac{h}{b} = \frac{\sqrt{R^2 - w^2}}{w}.$$

It follows that

$$\frac{hw}{\sqrt{L^2 - w^2}} + \frac{hw}{\sqrt{R^2 - w^2}} = w,$$

or,

$$\frac{1}{\sqrt{L^2 - w^2}} + \frac{1}{\sqrt{R^2 - w^2}} = \frac{1}{h}.$$

If $L = R$, it is trivial to solve this and find w in terms of L, R and h.

If $L \neq R$, our equation is harder to solve. One way of doing this is as follows. Let

$$\frac{1}{\sqrt{L^2 - w^2}} = \frac{1}{2h}(1 - x), \quad \frac{1}{\sqrt{R^2 - w^2}} = \frac{1}{2h}(1 + x).$$

Then

$$L^2 - w^2 = \frac{4h^2}{(1 - x)^2}, \quad R^2 - w^2 = \frac{4h^2}{(1 + x)^2}$$

and

$$L^2 - R^2 = 4h^2 \left(\frac{1}{(1 - x)^2} - \frac{1}{(1 + x)^2} \right) = \frac{16h^2 x}{(1 - x^2)^2}.$$

This can be written

$$\frac{x}{(1 - x)^2} = \frac{L^2 - R^2}{16h^2}.$$

From a graph, it is easy to see that this has a unique solution (no matter what the values of L, R and h) with $-1 < x < 1$.

Thus, for example, if $L = 3.9$, $R = 2.5$ and $h = \dfrac{9}{7}$,

$$\begin{aligned}\frac{x}{(1-x^2)^2} &= \frac{L^2-R^2}{16h^2} = \frac{3.9^2-2.5^2}{16\left(\dfrac{9}{7}\right)^2} = \frac{39^2-25^2}{16\left(\dfrac{90}{7}\right)^2} = \frac{49(39^2-25^2)}{16\times 90^2}\\ &= \frac{49\times 64\times 14}{16\times 8100} = \frac{2^7\times 7^3}{2^6\times 5^2\times 3^4} = \frac{2\times 7^3}{3^4\times 5^2} = \frac{2\times 7^3}{45^2}\\ &= \frac{2\times 7^3}{(49-4)^2} = \frac{2\times 7^3}{(7^2-2^2)^2} = \frac{\frac{2}{7}}{\left(1-\left(\frac{2}{7}\right)^2\right)^2},\end{aligned}$$

from which we see that $x = \dfrac{2}{7}$ and eventually, $w = \dfrac{3}{2}$.

In general, we have a quartic equation for x, which can be written

$$(L^2-R^2)(1-x^2)^2 - 16h^2x = 0,$$

or,

$$x^4 - 2x^2 - \frac{16h^2}{L^2-R^2}x + 1 = 0.$$

We can solve the quartic as follows. We will suppose $L > R$.

We suppose we can write the quartic

$$(x^2-A)^2 - (Bx+C)^2 = 0,$$

or,

$$x^4 - (2A+B^2)x^2 - 2BCx + (A^2-C^2) = 0,$$

where

$$2A + B^2 = 2, \quad 2BC = \frac{16h^2}{L^2-R^2}, \quad A^2 - C^2 = 1.$$

It follows that

$$B^2 = 2-2A, \quad C^2 = A^2-1 \text{ and } B^2C^2 = \frac{64h^4}{(L^2-R^2)^2},$$

so

$$(2-2A)(A^2-1) = \frac{64h^4}{(L^2-R^2)^2}.$$

This is a cubic in A, which can be written

$$A^3 - A^2 - A + 1 + \frac{64h^4}{(L^2 - R^2)^2} = 0.$$

To solve this cubic, we start by writing $A = y + \dfrac{1}{3}$ to remove the A^2 term, so

$$\left(y + \frac{1}{3}\right)^3 - \left(y + \frac{1}{3}\right)^2 - \left(y + \frac{1}{3}\right) + 1 + \frac{16h^4}{(L^2 - R^2)^2} = 0,$$

or,

$$y^3 - \frac{4}{3}y + \frac{16}{27} + \frac{16h^4}{(L^2 - R^2)^2} = 0.$$

If we draw a graph, we see that this has only one real root, so we solve it as follows.

We write the equation

$$y^3 = \frac{4}{3}y - \frac{16}{27} - \frac{16h^4}{(L^2 - R^2)^2}.$$

We now set $y = u + v$, and use the fact that $y^3 = 3uvy + (u^3 + v^3)$ to obtain

$$uv = \frac{4}{9}, \quad u^3 + v^3 = -\frac{16}{27} - \frac{16h^4}{(L^2 - R^2)^2}.$$

We have the sum of u^3 and v^3, and the product is $u^3v^3 = (uv)^3 = \dfrac{64}{729}$.

So u^3 and v^3 are the roots of the quadratic

$$z^2 + \left(\frac{16}{27} + \frac{16h^4}{(L^2 - R^2)^2}\right) z + \frac{64}{729} = 0,$$

or,

$$\left(z + \left(\frac{8}{27} + \frac{8h^4}{(L^2 - R^2)^2}\right)\right)^2 - \left(\frac{128h^4}{27(L^2 - R^2)^2} + \frac{64h^8}{(L^2 - R^2)^4}\right) = 0.$$

It follows that

$$
\begin{aligned}
u^3 &= -\frac{8}{27} - \frac{8h^4}{(L^2-R^2)^2} + \sqrt{\frac{128h^4}{27(L^2-R^2)^2} + \frac{64h^8}{(L^2-R^2)^4}} \\
&= -\frac{8}{27} - \frac{8h^4}{(L^2-R^2)^2} + \frac{8h^4}{(L^2-R^2)^2}\sqrt{1 + \frac{2(L^2-R^2)^2}{27h^4}}, \\
v^3 &= -\frac{8}{27} - \frac{8h^4}{(L^2-R^2)^2} - \frac{8h^4}{(L^2-R^2)^2}\sqrt{1 + \frac{2(L^2-R^2)^2}{27h^4}}, \\
u &= \left(-\frac{8}{27} - \frac{8h^4}{(L^2-R^2)^2} + \frac{8h^4}{(L^2-R^2)^2}\sqrt{1 + \frac{2(L^2-R^2)^2}{27h^4}}\right)^{\frac{1}{3}}, \\
v &= \left(-\frac{8}{27} - \frac{8h^4}{(L^2-R^2)^2} - \frac{8h^4}{(L^2-R^2)^2}\sqrt{1 + \frac{2(L^2-R^2)^2}{27h^4}}\right)^{\frac{1}{3}},
\end{aligned}
$$

$$
y = u + v, \quad A = y + \frac{1}{3}, \quad B = \sqrt{2 - 2A}, \quad C = \sqrt{A^2 - 1},
$$

and then we must solve

$$
(x^2 + A)^2 = (Bx + C)^2.
$$

This gives rise to the two quadratics

$$
x^2 - Bx + (A - C) = 0 \quad \text{and} \quad x^2 + Bx + (A + C) = 0.
$$

The first of these has a positive root, which is w. (Phew!)