**Mike's Maths Mysteries Problem 5**

**Solution**

$$I_k + I_{k-2} = \int_0^1 \frac{x^k + x^{k-2}}{x^2+1}\ dx = \int_0^1 x^{k-2}\ dx = \frac{1}{k-1}.$$

$$I_0 = \int_0^1 \frac{1}{x^2+1}\ dx = \left[\tan^{-1} x\right]_0^1 = \frac{\pi}{4},$$

$$I_1 = \int_0^1 \frac{x}{x^2+1}\ dx = \left[\frac{1}{2}\log(x^2+1)\right]_0^1 = \frac{1}{2}\log 2.$$

It follows that

$$I_2 = 1 - \frac{\pi}{4},\quad I_3 = \frac{1}{2} - \frac{1}{2}\log 2,\quad I_4 = \frac{1}{3} - 1 + \frac{\pi}{4},\quad I_5 = \frac{1}{4} - \frac{1}{2} + \frac{1}{2}\log 2,$$

$$I_6 = \frac{1}{5} - \frac{1}{3} + 1 - \frac{\pi}{4},\quad I_7 = \frac{1}{6} - \frac{1}{4} + \frac{1}{2} - \frac{1}{2}\log 2,\quad I_8 = \frac{1}{7} - \frac{1}{5} + \frac{1}{3} - 1 + \frac{\pi}{4},$$

and

$$\int_0^1 \frac{x^4(1-x)^4}{1+x^2}\ dx = I_4 - 4I_5 + 6I_6 - 4I_7 + I_8 = \frac{22}{7} - \pi.$$

Similarly

$$\int_0^1 \frac{x^8(1-x)^8}{1+x^2}\ dx = 4\left(\pi - \frac{43\times 1097}{3\times 5\times 7\times 11\times 13}\right) < \int_0^1 x^8(1-x)^8\ dx < 0.000005.$$

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