Mike's Maths Mysteries Problem 6

Solution

Let $y = f(x)$ be the quartic, and $y = s(x)$ be the secant through the inflections, with abscissae (x–coordinates) a and b . Then $f(x) - s()x)$ has zeroes at a and b, so

$$f(x) - s(x) = K(x-a)(x-b)(x^2 + cx + d)$$

for some K, c and d.

That is,

$$\begin{aligned} f(x) - s(x) &= K(x^2 - (a+b)x + ab)(x^2 + cx + d) \\ &= K(x^4 - (a+b-c)x^3 + (ab - (a+b)c + d)x^2 + (abc - (a+b)d)x + abd). \end{aligned}$$

It follows by differentiating twice, and remembering that $s''(x) = 0$, that

$$f''(x) = K(12x^2 - 6(a+b-c)x + 2(ab - (a+b)c + d).$$

Now $f''(a) = 0, \quad f''(b) = 0$, so

$$12a^2 - 6(a+b-c)a^2 + 2(ab-(a+b)c+d) = 0 \text{ and } 12b^2 - 6(a+b-c)b + 2(ab-(a+b)c+d) = 0.$$

That is,

$$(2a-b)c + d = -3a^2 + 2ab, \quad (2b-a)c + d = -3b^2 + 2ab.$$

It follows that

$$c = -(a+b), \quad d = -a^2 + 3ab - b^2,$$

and

$$\begin{aligned} f(x) - s(x) &= K(x-a)(x-b)(x^2 - (a+b)x - (a^2 - 3ab + b^2)) \\ &= K(x-a)(x-b)(x - (\alpha a + \beta b))(x - (\beta a + \alpha b)), \end{aligned}$$

where

$$\alpha = \frac{1+\sqrt{5}}{2}, \quad \beta = \frac{1-\sqrt{5}}{2}.$$

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It follows that the abscissae of the four points are, in order,

$$\alpha a + \beta b, \quad a, \quad b, \quad \beta a + \alpha b,$$

and the three distances between the abscissae are

$$\frac{\sqrt{5}-1}{2}|b-a|, \quad |b-a|, \quad \frac{\sqrt{5}-1}{2}|b-a|,$$

in the ratio

$$1 : \phi : 1,$$

where ϕ is the Golden Ratio, $\phi = \dfrac{1+\sqrt{5}}{2}$.