Mike's Maths Mysteries Problem 7

Solution

First of all,

$$\begin{aligned} x^{10}+x^8+x^6+x^4+x^2+1 &= (x^2+1)(x^8+x^4+1) \\ &= (x^2+1)((x^4+1)^2-(x^2)^2) \\ &= (x^2+1)(x^4+x^2+1)(x^4-x^2+1) \\ &= (x^2+1)((x^2+1)^2-x^2)((x^2+1)^2-(\sqrt{3}x)^2) \\ &= (x^2+1)(x^2+x+1)(x^2-x+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1). \end{aligned}$$

(This can also be obtained from deMoivre's Theorem.)

Similarly,

$$\begin{aligned} x^5+x^4+x^3+x^2+x+1 = (x+1)(x^4+x^2+1) &= (x+1)((x^2+1)^2-x^2) \\ &= (x+1)(x^2+x+1)(x^2-x+1) \end{aligned}$$

and

$$\begin{aligned} x^4+x^3+x^2+x+1 &= (x^2+\frac{1}{2}x+1)^2-\frac{5}{4}x^2 = (x^2+\frac{1+\sqrt{5}}{2}x+1)(x^2+\frac{1-\sqrt{5}}{2}x+1) \\ &= (x^2+\alpha x+1)(x^2+\beta x+1), \end{aligned}$$

where

$$\alpha=\frac{1+\sqrt{5}}{2}, \quad \beta=\frac{1-\sqrt{5}}{2},$$

as in MMM6.

Incidentally, deMoivre's Theorem gives us

$$x^4+x^3+x^2+x+1=(x^2-2\cos\frac{2\pi}{5}x+1)(x^2-2\cos\frac{4\pi}{5}x+1),$$

from which we decuce that

$$\cos\frac{2\pi}{5}=-\frac{\beta}{2}, \quad \cos\frac{4\pi}{5}=-\frac{\alpha}{2}.$$

It follows that

$$
\begin{aligned}
&\int_0^1 \frac{1}{1+x^2+x^4+x^6+x^8+x^{10}}\ dx \\
&= \int_0^1 \frac{1}{(x^2+1)(x^2+x+1)(x^2-x+1)(x^2+\sqrt{3}x+1)(x^2-\sqrt{3}x+1)}\ dx \\
&= \int_0^1 \frac{1}{3}\frac{1}{x^2+1} + \frac{1}{4}\frac{x+1}{x^2+x+1} - \frac{1}{4}\frac{x-1}{x^2-x+1} + \frac{1}{12}\frac{\sqrt{3}x+1}{x^2+\sqrt{3}x+1} - \frac{1}{12}\frac{\sqrt{3}x-1}{x^2-\sqrt{3}x+1}\ dx \\
&= \int_0^1 \frac{1}{3}\frac{1}{x^2+1} + \frac{1}{4}\frac{\frac{1}{2}(2x+1)+\frac{1}{2}}{x^2+x+1} - \frac{1}{4}\frac{\frac{1}{2}(2x-1)-\frac{1}{2}}{x^2-x+1} \\
&\qquad\qquad + \frac{1}{12}\frac{\frac{\sqrt{3}}{2}(2x+\sqrt{3})-\frac{1}{2}}{x^2+\sqrt{3}x+1} - \frac{1}{12}\frac{\frac{\sqrt{3}}{2}(2x-\sqrt{3})+\frac{1}{2}}{x^2-\sqrt{3}x+1}\ dx \\
&= \Big[\frac{1}{3}\tan^{-1} x + \frac{1}{8}\log(x^2+x+1) - \frac{1}{8}\log(x^2-x+1) \\
&\qquad + \frac{\sqrt{3}}{24}\log(x^2+\sqrt{3}x+1) - \frac{\sqrt{3}}{24}\log(x^2-\sqrt{3}x+1)\Big]_0^1 \\
&\qquad + \int_0^1 \frac{1}{8}\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} + \frac{1}{8}\frac{1}{(x-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \\
&\qquad\qquad - \frac{1}{24}\frac{1}{(x+\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2} - \frac{1}{24}\frac{1}{(x-\frac{\sqrt{3}}{2})^2+(\frac{1}{2})^2}\ dx \\
&= \Big[\frac{1}{3}\tan^{-1} x + \frac{1}{8}\log(x^2+x+1) - \frac{1}{8}\log(x^2-x+1) \\
&\qquad + \frac{\sqrt{3}}{24}\log(x^2+\sqrt{3}x+1) - \frac{\sqrt{3}}{24}\log(x^2-\sqrt{3}x+1) \\
&\qquad + \frac{\sqrt{3}}{12}\tan^{-1}\frac{2x+1}{\sqrt{3}} + \frac{\sqrt{3}}{12}\tan^{-1}\frac{2x-1}{\sqrt{3}} \\
&\qquad - \frac{1}{12}\tan^{-1}(2x+\sqrt{3}) - \frac{1}{12}\tan^{-1}(2x-\sqrt{3})\Big]_0^1 \\
&= \frac{1}{3}\left(\frac{\pi}{4}-0\right) + \frac{1}{8}(\log 3 - 0) - \frac{1}{8}(0-0) + \frac{\sqrt{3}}{24}\left(\log(2+\sqrt{3})-0\right) \\
&\qquad - \frac{\sqrt{3}}{24}\left(\log(2-\sqrt{3})-0\right) + \frac{\sqrt{3}}{12}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) + \frac{\sqrt{3}}{12}\left(\frac{\pi}{6}-\left(-\frac{\pi}{6}\right)\right) \\
&\qquad - \frac{1}{12}\left(\frac{5\pi}{12}-\frac{\pi}{3}\right) - \frac{1}{12}\left(\frac{\pi}{12}-\left(-\frac{\pi}{3}\right)\right)
\end{aligned}
$$

$$= \frac{(1+\sqrt{3})\pi}{24} + \frac{\log 3}{8} + \frac{\sqrt{3}\log(2+\sqrt{3})}{12} = \frac{(1+\sqrt{3})\pi + \log 27 + \sqrt{12}\log(2+\sqrt{3})}{24}.$$

You may be wondering how I did the partial fractions. I have a quicker method than you were taught in school. I use MAPLE, and the command "convert(f,parfrac,x)". But you must give MAPLE the factorised form of the denominator!

$$\int_0^1 \frac{1}{1+x+x^2+x^3+x^4+x^5}\,dx = \int_0^1 \frac{1}{(x+1)(x^2+x+1)(x^2-x+1)}\,dx$$
$$= \int_0^1 \frac{1}{3}\frac{1}{x+1} + \frac{1}{2}\frac{1}{x^2+x+1} - \frac{1}{6}\frac{2x-1}{x^2-x+1}\,dx$$
$$= \int_0^1 \frac{1}{3}\frac{1}{x+1} + \frac{1}{2}\frac{1}{(x+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{6}\frac{2x-1}{x^2-x+1}\,dx$$
$$= \Big[\frac{1}{3}\log(x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}} - \frac{1}{6}\log(x^2-x+1)\Big]_0^1$$
$$= \frac{1}{3}(\log 2 - 0) + \frac{1}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right) - \frac{1}{6}(0-0)$$
$$= \frac{\log 2}{3} + \frac{\pi}{6\sqrt{3}} = \frac{\pi\sqrt{3}+\log 64}{18}.$$

That was easier!

$$\int_0^1 \frac{1}{1+x+x^2+x^3+x^4}\,dx = \int_0^1 \frac{1}{(x^2+\alpha x+1)(x^2+\beta x+1)}\,dx$$
$$= \int_0^1 \frac{1}{\sqrt{5}}\frac{x+\alpha}{x^2+\alpha x+1} - \frac{1}{\sqrt{5}}\frac{x+\beta}{x^2+\beta x+1}\,dx$$
$$= \int_0^1 \frac{1}{2\sqrt{5}}\frac{2x+\alpha}{x^2+\alpha x+1} - \frac{\alpha}{2\sqrt{5}}\frac{1}{x^2+\alpha x+1}$$
$$\qquad - \frac{1}{2\sqrt{5}}\frac{2x+\beta}{x^2+\beta x+1} + \frac{\beta}{2\sqrt{5}}\frac{1}{x^2+\beta x+1}\,dx$$
$$= \Big[\frac{1}{2\sqrt{5}}\log(x^2+\alpha x+1) - \frac{1}{2\sqrt{5}}\log(x^2+\beta x+1)\Big]_0^1$$
$$\qquad - \frac{\alpha}{2\sqrt{5}}\int_0^1 \frac{1}{(x+\frac{\alpha}{2})^2+(\frac{\sqrt{10-2\sqrt{5}}}{4})^2} + \frac{\beta}{2\sqrt{5}}\frac{1}{(x+\frac{\beta}{2})^2+(\frac{\sqrt{10+2\sqrt{5}}}{4})^2}\,dx$$

$$
\begin{aligned}
&= \Big[\frac{1}{2\sqrt{5}} \log(x^2 + \alpha x + 1) - \frac{1}{2\sqrt{5}} \log(x^2 + \beta x + 1) \\
&\qquad - \frac{\alpha}{2\sqrt{5}} \cdot \frac{\alpha}{\sqrt{5}} \sqrt{10 - 2\sqrt{5}} \tan^{-1} \left(\frac{\alpha}{\sqrt{5}} \sqrt{10 - 2\sqrt{5}} \left(x + \frac{\alpha}{2} \right) \right) \\
&\qquad + \frac{\beta}{2\sqrt{5}} \cdot - \frac{\beta}{\sqrt{5}} \sqrt{10 + 2\sqrt{5}} \tan^{-1} \left(- \frac{\beta}{\sqrt{5}} \sqrt{10 + 2\sqrt{5}} \left(x + \frac{\beta}{2} \right) \right) \Big]_0^1 \\
&= \Big[\frac{1}{2\sqrt{5}} \log(x^2 + \alpha x + 1) - \frac{1}{2\sqrt{5}} \log(x^2 + \beta x + 1) \\
&\qquad - \frac{\alpha^2}{10} \sqrt{10 - 2\sqrt{5}} \tan^{-1} \left(\frac{\alpha}{\sqrt{5}} \sqrt{10 - 2\sqrt{5}} \left(x + \frac{\alpha}{2} \right) \right) \\
&\qquad - \frac{\beta^2}{10} \sqrt{10 + 2\sqrt{5}} \tan^{-1} \left(- \frac{\beta}{\sqrt{5}} \sqrt{10 + 2\sqrt{5}} \left(x + \frac{\beta}{2} \right) \right) \Big]_0^1
\end{aligned}
$$

etc.