Mike's Monthly Maths Mysteries Problem 1 Solution

I. 1.

Let $\dfrac{x}{y} = r$. Then we have

$$\begin{aligned}
2\left(r+1+\frac{1}{r}\right)^2\left(\frac{1}{2r}-1+\frac{r}{2}\right) &= \left(r+1+\frac{1}{r}\right)^2\left(r-2+\frac{1}{r}\right) = \\
&= \left(r^2+2r+3+\frac{2}{r}+\frac{1}{r^2}\right)\left(r-2+\frac{1}{r}\right) \\
&= r^3+0r^2+0r-2+\frac{0}{r}+\frac{0}{r^2}+\frac{1}{r^3} \\
&= \frac{x^3}{y^3}-2+\frac{y^3}{x^3}.
\end{aligned}$$

2.

$$\begin{aligned}
\frac{x^6-a^3x^3-2a^6}{\dfrac{x^2}{a^2}-\dfrac{x}{a}+1} &= a^2\frac{x^6-a^3x^3-2a^6}{x^2-ax+a^2} \\
&= a^2(x^4+ax^3-2x^3x-2a^4) \quad \text{by long division} \\
&= a^2\left(x^3(x+a)-2a^3(x+a)\right) \\
&= a^2(x^3-2a^3)(x+a).
\end{aligned}$$

3.

$$\begin{aligned}
\left(\frac{x}{x-a}\right)^2+\left(\frac{x}{x+a}\right)^2 &= x^2\left(\frac{1}{(x-a)^2}+\frac{1}{(x+a)^2}\right) \\
&= x^2\left(\frac{(x+a)^2+(x-a)^2}{(x^2-a^2)^2}\right) \\
&= \frac{2x^2(x^2+a^2)}{(x^2-a^2)^2}
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{2\dfrac{x^2}{a^2}\left(\dfrac{x^2}{a^2}+1\right)}{\left(\dfrac{x^2}{a^2}-1\right)^2} \\
&= \frac{2\left(\dfrac{n+1}{n-1}\right)\left(\dfrac{n+1}{n-1}+1\right)}{\left(\dfrac{n+1}{n-1}-1\right)^2} \\
&= \frac{2(n+1)\left((n+1)+(n-1)\right)}{\left((n+1)-(n-1)\right)^2} \\
&= \frac{2(n+1)(2n)}{2^2} \\
&= n(n+1).
\end{aligned}
$$

II. 1.

$$
\begin{aligned}
(a^2b^2-1)(x^2-y^2)+4abxy &= (a^2b^2-1)x^2+4abxy-(a^2b^2-1)y^2 \\
&= (ab+1)(ab-1)x^2+4abxy-(ab+1)(ab-1)y^2 \\
&= \left((ab+1)x-(ab-1)y\right)\left((ab-1)x+(ab+1)y\right).
\end{aligned}
$$

2.

$$
\begin{aligned}
216x^6+19x^3-1 &= (27x^3-1)(8x^3+1) \\
&= (3x-1)(9x^2+3x+1)(2x+1)(4x^2-2x+1).
\end{aligned}
$$

III.

By division,

$$18a^5-45a^4-5a-14 = \frac{2}{3}(27a^5-45a^4-16)-(15a^4+5a+\frac{10}{3}),$$

$$27a^5-45a^4-16 = (\frac{9}{5}a-3)(15a^4+5a+\frac{10}{3})-(9a^2-9a+6),$$

$$15a^4 + 5a + \frac{10}{3} = (\frac{5}{3}a^2 + \frac{5}{3}a + \frac{5}{9})(9a^2 - 9a + 6).$$

So the highest common factor is some multiple of $9a^2 - 9a + 6$.

Indeed,

$$18a^5 - 45a^4 - 5a - 14 = (3a^2 - 3a + 2)(6a^3 - 9a^2 - 13a - 7)$$

and

$$27a^5 - 45a^4 - 16 = (3a^2 - 3a + 2)(9a^3 - 6a^2 - 12a - 8).$$

IV.

1. Let

$$a = x + y - z, \quad b = y + z - x, \quad c = z + x - y.$$

Then

$$a + b + c = x + y + z \text{ and } a = x + y - z,$$

so, by subtraction,

$$z = \frac{1}{2}(b + c)$$

and similarly

$$y = \frac{1}{2}(a + b) \text{ and } x = \frac{1}{2}(c + a).$$

Then

$$x(y + z - x)^2 + y(z + x - y)^2 + z(x + y - z)^2 + (x + y - z)(y + z - x)(z + x - y)$$
$$= \frac{1}{2}(c + a)b^2 + \frac{1}{2}(a + b)c^2 + \frac{1}{2}(b + c)a^2 + abc$$
$$= \frac{1}{2}(ab^2 + ac^2 + ba^2 + bc^2 + ca^2 + cb^2 + 2abc)$$
$$= \frac{1}{6}(3a^2b + 3ab^2 + 3b^2c + 3b^2c + 3c^2a + 3ca^2 + 6abc)$$
$$= \frac{1}{6}\left((a + b + c)^3 - a^3 - b^3 - c^3\right)$$

$$= \frac{1}{6}\left((b+c)\left((a+b+c)^2 + a(a+b+c) + a^2\right) - (b+c)\left(b^2 - bc + c^2\right)\right)$$
$$= \frac{1}{6}(b+c)\left(a^2 + 2ab + 2ac + 2bc + b^2 + c^2 + a^2 + ab + ac + a^2 - b^2 + bc - c^2\right)$$
$$= \frac{1}{6}(b+c)\left(3a^2 + 3ab + 3ac + 3bc\right)$$
$$= \frac{1}{2}(b+c)\left(a^2 + ab + ac + bc\right)$$
$$= \frac{1}{2}(b+c)(a+b)(a+c)$$
$$= 4xyz.$$

2.

$$\left(\frac{x+4}{x^2-x-12} - \frac{x+3}{x^2+x-12}\right) \Big/ \left(1 + \frac{2(x^2-12)}{x^2+7x+12}\right)$$
$$= \left(\frac{x+4}{(x-4)(x+3)} - \frac{x+3}{(x+4)(x-3)}\right) \Big/ \left(1 + \frac{2(x^2-12)}{(x+3)(x+4)}\right)$$
$$= \frac{(x+4)^2(x-3) - (x+3)^2(x-4)}{(x^2-16)(x^2-9)} \Big/ \frac{3x^2+7x-12}{(x+3)(x+4)}$$
$$= \frac{(x^2+8x+16)(x-3) - (x^2+6x+9)(x-4)}{(x-4)(x-3)(3x^2+7x-12)}$$
$$= \frac{(x^3+5x^2-8x-48) - (x^3+2x^2-15x-36)}{(x-4)(x-3)(3x^2+7x-12)}$$
$$= \frac{3x^2+7x-12}{(x-4)(x-3)(3x^2+7x-12)}$$
$$= \frac{1}{(x-4)(x-3)}.$$

V.

$$(x^2+y^2-2)^2 + 4(xy+2)(x^2+xy+y^2)$$
$$= x^4 + y^4 + 4 - 4x^2 - 4y^2 + 2x^2y^2 + 4x^3y + 4x^2y^2 + 4xy^3 + 8x^2 + 8xy + 8y^2$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 + 4x^2 + 8xy + 4y^2 + 4$$
$$= (x+y)^4 + 4(x+y)^2 + 4$$
$$= \left((x+y)^2 + 2\right)^2 .$$

So the square root of the left side is

$$(x+y)^2 + 2.$$

VI.

1.
$$\frac{2x^2 - x - 1}{2x - 1} + \frac{6x^2 - 4x + 1}{3x - 2} = \frac{2}{6x - 13} + \frac{6x^2 - 9x - 1}{2x - 3}$$

can be written

$$x - \frac{1}{2x-1} + 2x + \frac{1}{3x-2} = \frac{2}{6x-13} + 3x - \frac{1}{2x-3},$$

$$\frac{1}{2x-3} - \frac{1}{2x-1} = \frac{2}{6x-13} - \frac{1}{3x-2} = \frac{2}{6x-13} - \frac{2}{6x-4},$$

$$\frac{2}{(2x-1)(2x-3)} = \frac{18}{(6x-13)(6x-4)},$$

$$\frac{18}{(6x-3)(6x-9)} = \frac{18}{(6x-13)(6x-4)},$$

$$(6x-3)(6x-9) = (6x-13)(6x-4),$$

$$36x^2 - 72x + 27 = 36x^2 - 102x + 52,$$

$$30x = 25,$$

$$x = \frac{5}{6}.$$

2.
$$2\frac{3}{5} + \frac{3x - 5y}{2} = \frac{2}{5}(x+2), \quad 8 - \frac{x - 2y}{4} = \frac{x}{2} + \frac{y}{3}$$

are equivalent to

$$26 + 5(3x - 5y) = 4(x + 2), \quad 96 - 3(x - 2y) = 6x + 4y,$$

$$26 + 15x - 25y = 4x + 8, \quad 96 - 3x + 6y = 6x + 4y,$$

$$11x - 25y = -18, \quad 9x - 2y = 96,$$

$$22x - 50y = -36, \quad 225x - 50y = 2400,$$

from which we find

$$203x = 2436, \quad x = 12, \quad y = 6.$$

3.

$$ab(x^2 + 1) = x(a^2 + b^2)$$

is

$$abx^2 - (a^2 + b^2)x + ab = 0,$$

$$(ax - b)(bx - a) = 0,$$

$$x = \frac{b}{a} \text{ or } \frac{a}{b}.$$

VII.

Let B's income be x Rupees. Then A's income is $\frac{7}{5}x$ Rupees.

B's expenditure is 32 Rupees, so his savings are $x - 32$ Rupees.

A's expenditure is $3(x - 32)$ Rupees, so A's savings are

$$\frac{7}{5}x - 3(x - 32) = 96 - \frac{8}{5}x \text{ Rupees.}$$

A's new income is $\frac{10}{7} \times \frac{7x}{5} = 2x$ Rupees and his new expenditure is $\frac{7}{6} \times 3(x - 32) = \frac{7}{2}(x - 32)$ Rupees, so his new savings are

$$2x - \frac{7}{2}(x - 32) = 112 - \frac{3}{2}x \text{ Rupees.}$$

So we have

$$112 - \frac{3}{2}x - (96 - \frac{8}{5}x) = 21,$$

or,

$$\frac{1}{10}x = 21 + 96 - 112 = 5,$$

$$x = 50.$$