Solutions to MMMM2

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$$\begin{aligned}
&\sin^4\left(x+\frac{\pi}{4}\right)-\sin^4\left(x-\frac{\pi}{4}\right)\\
&=\left(\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right)^4-\left(\sin x\cos\frac{\pi}{4}-\sin x\cos\frac{\pi}{4}\right)^4\\
&=\left(\frac{\sin x}{\sqrt{2}}+\frac{\cos x}{\sqrt{2}}\right)^4-\left(\frac{\sin x}{\sqrt{2}}-\frac{\cos x}{\sqrt{2}}\right)^4\\
&=\frac{1}{4}\left(\left(\sin x+\cos x\right)^4-\left(\sin x-\cos x\right)^4\right)\\
&=\frac{1}{4}\left(8\sin^3 x\cos x+8\sin x\cos^3 x\right)\\
&=2\sin x\cos x\left(\sin^2 x+\cos^x\right)\\
&=\sin(2x).
\end{aligned}$$

$$\begin{aligned}
&\sin^6\left(x+\frac{\pi}{6}\right)+\cos^6\left(x+\frac{\pi}{6}\right)-\sin^6\left(x-\frac{\pi}{6}\right)-\cos^6\left(x-\frac{\pi}{6}\right)\\
&=\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)^6+\left(\frac{\sqrt{3}}{2}\cos x-\frac{1}{2}\sin x\right)^6-\left(\frac{\sqrt{3}}{2}\sin x-\frac{1}{2}\cos x\right)^6\\
&\qquad-\left(\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x\right)^6\\
&=\frac{1}{64}\left(\left(\sqrt{3}\sin x+\cos x\right)^6+\left(\sqrt{3}\cos x-\sin x\right)^6-\left(\sqrt{3}\sin x-\cos x\right)^6\right.\\
&\qquad\left.-\left(\sqrt{3}\cos x+\sin x\right)^6\right)\\
&=\frac{1}{64}\left((\sqrt{3}s+c)^6+(\sqrt{3}c-s)^6-(\sqrt{3}s-c)^6-(\sqrt{3}c+s)^6\right)
\end{aligned}$$

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\begin{aligned}
&= \frac{1}{64}\left(\left(27s^6 + 54\sqrt{3}s^5c + 135s^4c^2 + 60\sqrt{3}s^3c^3 + 45s^2c^4 + 6\sqrt{3}sc^5 + c^6\right)\right.\\
&\qquad +\left(27c^6 - 54\sqrt{3}c^5s + 135c^4s^2 - 60\sqrt{3}c^3s^3 + 45c^2s^4 - 6\sqrt{3}cs^5 + s^6\right)\\
&\qquad -\left(27s^6 - 54\sqrt{3}s^5c + 135s^4c^2 - 60\sqrt{3}s^3c^3 + 45s^2c^4 - 6\sqrt{3}sc^5 + c^6\right)\\
&\qquad \left.-\left(27c^6 + 54\sqrt{3}c^5s + 135c^4s^2 + 60\sqrt{3}c^3s^3 + 45c^2s^4 + 6\sqrt{3}cs^5 + s^6\right)\right)\\
&= \frac{1}{64}\left(96\sqrt{3}s^5c - 96\sqrt{3}sc^5\right)\\
&= \frac{3\sqrt{3}}{2}sc(s^4 - c^4)\\
&= \frac{3\sqrt{3}}{2}sc(s^2 - c^2)(s^2 + c^2)\\
&= -\frac{3\sqrt{3}}{4}\sin(2x)\cos(2x)\\
&= -\frac{3\sqrt{3}}{8}\sin(4x).
\end{aligned}
$$

$$2\cos^2\frac{\pi}{8} - 1 = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

so

$$\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2} = \sin\frac{3\pi}{8},$$

and

$$\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2} = \cos\frac{3\pi}{8}.$$

Let

$$a = \frac{\sqrt{2+\sqrt{2}}}{2}, \quad b = \frac{\sqrt{2-\sqrt{2}}}{2}.$$

Note that

$$ab = \frac{\sqrt{2+\sqrt{2}}}{2} \cdot \frac{\sqrt{2-\sqrt{2}}}{2} = \frac{\sqrt{4-2}}{4} = \frac{1}{2\sqrt{2}},$$

$$a^2 - b^2 = \frac{2+\sqrt{2}}{4} - \frac{2-\sqrt{2}}{4} = \frac{1}{\sqrt{2}}$$

and

$$a^6 - b^6 = \frac{(2+\sqrt{2})^3}{64} - \frac{(2-\sqrt{2})^3}{64} = \frac{28\sqrt{2}}{64} = \frac{7}{8\sqrt{2}}.$$

Then

$$\begin{aligned}
&\sin^8\left(x+\frac{\pi}{8}\right) - \sin^8\left(x+\frac{3\pi}{8}\right) + \sin^8\left(x+\frac{5\pi}{8}\right) - \sin^8\left(x+\frac{7\pi}{8}\right)\\
&= (as+bc)^8 - (bs+ac)^8 + (-bs+ac)^8 - (-as+bc)^8\\
&= a^8s^8 + 8a^7bs^7c + 28a^6b^2s^6c^2 + 56a^5b^3s^5c^3 + 70a^4b^4s^4c^4\\
&\qquad + 56a^3b^5s^3c^5 + 28a^2b^6s^2c^6 + 8ab^7sc^7 + b^8c^8\\
&- b^8s^8 - 8ab^7s^7c - 28a^2b^6s^6c^2 - 56a^3b^5s^5c^3 - 70a^4b^4s^4c^4\\
&\qquad - 56a^5b^3s^3c^5 - 28a^6b^2s^2c^6 - 8a^7bsc^7 - a^8c^8\\
&+ b^8s^8 - 8ab^7s^7c + 28a^2b^6s^6c^2 - 56a^3b^5s^5c^3 + 70a^4b^4s^4c^4\\
&\qquad - 56a^5b^3s^3c^5 + 28a^6b^2s^2c^6 - 8a^7bsc^7 + a^8c^8\\
&- a^8s^8 + 8a^7bs^7c - 28a^6b^2s^6c^2 + 56a^5b^3s^5c^3 - 70a^4b^4s^4c^4\\
&\qquad + 56a^3b^5s^3c^5 - 28a^2b^6s^2c^6 + 8ab^7sc^7 - b^8c^8\\
&= (16a^7b - 16ab^7)s^7c + (112a^5b^3 - 112a^3b^5)s^5c^3\\
&\qquad - (112a^5b^3 - 112a^3b^5)s^3c^5 - (16a^7b - 16ab^7)sc^7\\
&= 16ab(a^6 - b^6)s^7b + 112a^3b^3(a^2 - b^2)s^5c^3\\
&\qquad - 112a^3b^3(a^2 - b^2)s^3c^5 - 16ab(a^6 - b^6)sc^7\\
&= \frac{7}{2}\left(s^7c + s^5c^3 - s^3c^5 - sc^7\right)
\end{aligned}$$

$$= \frac{7}{2}\left(sc(s^6 - c^6) + s^3c^3(s^2 - c^2)\right)$$
$$= \frac{7}{2}sc(s^2 - c^2)(s^4 + 2s^2c^2 + c^4)$$
$$= \frac{7}{2}sc(s^2 - c^2)(s^2 + c^2)^2$$
$$= -\frac{7}{4}\sin(2x)\cos(2x)$$
$$= -\frac{7}{8}\sin(4x).$$

$$2\cos^2\frac{\pi}{12} - 1 = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2},$$

so

$$\cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2} = \sin\frac{5\pi}{12}$$

and

$$\sin\frac{\pi}{12} = \frac{\sqrt{2-\sqrt{3}}}{2} = \cos\frac{5\pi}{12}.$$

Let

$$a = \frac{\sqrt{2+\sqrt{3}}}{2}, \quad b = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

Then

$$\sin^{12}\left(x+\frac{\pi}{12}\right) - \sin^{12}\left(x+\frac{3\pi}{12}\right) + \sin^{12}\left(x+\frac{5\pi}{12}\right) - \sin^{12}\left(x+\frac{7\pi}{12}\right)$$
$$+ \sin^{12}\left(x+\frac{9\pi}{12}\right) - \sin^{12}\left(x+\frac{11\pi}{12}\right)$$
$$= (as+bc)^{12} - \left(\frac{s+c}{\sqrt{2}}\right)^{12} + (bs+ac)^{12} - (-bs+ac)^{12} + \left(\frac{-s+c}{\sqrt{2}}\right)^{12} - (-as+bc)^{12}$$
$$= \frac{165}{128}cs(c^2-3s^2)(3c^2-s^2)(c^2+s^2)^2$$
$$= \frac{165}{128}cs(c^2-3s^2)(3c^2-s^2)$$

$$= \frac{165}{256}\sin(6x),$$

since

$$\sin(6x) = \frac{1}{2i}\left((c+is)^6 - (c-is)^6\right) = 2cs(c^2 - 3s^2)(3c^2 - s^2).$$

(OK, so I used the computer to do the last few steps. Forgive me.)

Total time taken: less than three hours.