Factorisations of certain identities of Ramanujan

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§1 Introduction

I want to describe to you some of the amazing and beautiful identities stated by Ramanujan in his Notebooks, including the Lost Notebook. In order to do so, I intend to use some modern notation, which simplifies the presentation of these identities.

Thus,

$$(a;q)_n = \begin{cases} 1 & \text{if } n=0 \\ (1-a)(1-aq)\ \cdots\ (1-aq^{n-1}) & \text{for } n \geq 1. \end{cases}$$

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)\ \cdots\ ,$$

$$(a_1, a_2,\ \cdots\ , a_n; q)_\infty = (a_1;q)_\infty (a_2;q)_\infty\ \cdots\ (a_n;q)_\infty,$$

$$\left(\begin{matrix} a_1, a_2,\ \cdots\ , a_m \\ b_1, b_2,\ \cdots\ , b_n \end{matrix}; q\right)_\infty = \frac{(a_1, a_2,\ \cdots\ , a_m; q)_\infty}{(b_1, b_2,\ \cdots\ , b_n; q)_\infty}.$$

One of Ramanujan's identities, considered by MacMahon and Hardy as perhaps Ramanujan's most beautiful identity is as follows.

The partition numbers, $p(n)$, are generated by

$$\sum_{n=0}^{\infty} p(n) q^n = \frac{1}{(q;q)_\infty}.$$

Then Ramanujan proved that

$$\sum_{n=0}^{\infty} p(5n+4)q^n = 5\,\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6}. \tag{1}$$

Quite apart from its stunning simplicity, this identity proves that

$$p(5n+4) \;\equiv\; 0 \pmod 5.$$

On the way to proving (1), we will prove Ramanujan's 5–dissection of Euler's product

$$(q;q)_\infty = (q^{25};q^{25})_\infty \left\{ R(q^5)^{-1} - q - q^2 R(q^5) \right\}, \tag{2}$$

where

$$R(q) = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix} ; q^5 \right)_\infty .$$

All this will be explained as we move along, so don't feel overwhelmed!

Incidentally, $R(q)$ plays a really important role in some of Ramanujan's work, since, as he discovered,

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\;\cdots}}}}} = R(q). \tag{3}$$

This is known as "the Rogers–Ramanujan continued fraction". It was first discovered by L. J. Rogers in the 1890's, and rediscovered by Ramanujan before he went to England. It does NOT, as is often stated, depend upon two famous identities known as the Rogers–Ramanujan identities. If time permits, I will derive it.

Ramanujan considered the quantity

$$k = qR(q)R(q^2)^2 = q \left(\begin{matrix} q, q^2, q^8, q^9 \\ q^3, q^4, q^6, q^7 \end{matrix} ; q^{10} \right)_\infty , \tag{4}$$

aand stated a number of facts relating to k, including

$$\frac{1}{k} - k = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5}. \tag{5}$$

I believe the following two identities involving k were discovered by Shaun Cooper. He certainly proved them (2009).

$$\frac{1}{k}+1-k=\frac{(q^2;q^2)_\infty^4(q^5;q^5)_\infty^2}{q(q;q)_\infty^2(q^{10};q^{10})_\infty^4} \tag{6}$$

and

$$\frac{1}{k}-4-k=\frac{(q;q)_\infty^3(q^5;q^5)_\infty}{q(q^2;q^2)_\infty(q^{10};q^{10})_\infty^3}. \tag{7}$$

It is my intention to prove at least one of these in this talk.

§2 Jacobi's triple product identity

Jacobi's triple product identity can be written (here $|q|<1$ and $a\neq 0$),

$$(-a^{-1}q,-aq,q^2;q^2)_\infty=\sum_{k=-\infty}^{\infty}a^kq^{k^2}.$$

A simple proof might proceed as follows.

We can start with an identity equivalent to one of Euler's.

$$(-aq;q^2)_\infty=\sum_{k=0}^{\infty}a^k\frac{q^{k^2}}{(q^2;q^2)_k}.$$

By induction on n,

$$(-a^{-1}q;q^2)_n(-aq;q^2)_\infty=\sum_{k=-n}^{\infty}a^k\frac{q^{k^2}}{(q^2;q^2)_{n+k}}.$$

Now let $n\to\infty$, and multiply by $(q^2;q^2)_\infty$ to obtain Jacobi's triple product identity.

Jacobi's triple product identity can be written in the equivalent form

$$(aq,a^{-1}q^2,q^3;q^3)_\infty=\sum_{k=-\infty}^{\infty}(-1)^ka^kq^{(3k^2-k)/2}$$

of which the special case $a=1$ is a celebrated identity of Euler,

$$(q;q)_\infty=\sum_{k=-\infty}^{\infty}(-1)^kq^{(3k^2-k)/2}. \tag{8}$$

Two further special cases of importance are

$$\sum_{-\infty}^{\infty}(-1)^k q^{(5k^2-3k)/2} = (q, q^4, q^5; q^5)_\infty \tag{9}$$

and

$$\sum_{-\infty}^{\infty}(-1)^k q^{(5k^2-k)/2} = (q^2, q^3, q^5; q^5)_\infty. \tag{10}$$

Also equivalent to Jacobi's triple product identity is

$$(a^{-1}, aq, q; q)_\infty = \sum_{k=0}^{\infty}(-1)^k (a^k - a^{-k-1}) q^{(k^2+k)/2}. \tag{11}$$

We can divide by $(1 - a^{-1})$ (provided $a \neq 1$) and obtain

$$\begin{aligned}(a^{-1}q, aq, q; q)_\infty &= \sum_{k=0}^{\infty}(-1)^k \frac{a^{k+\frac{1}{2}} - a^{-(k+\frac{1}{2})}}{a^{\frac{1}{2}} - a^{-\frac{1}{2}}} q^{(k^2+k)/2} \\ &= 1 + \sum_{k=1}^{\infty}(-1)^k (a^k + a^{k-1} + \cdots + a^{-k}) q^{(k^2+k)/2}.\end{aligned} \tag{12}$$

If we let $a \to 1$ in this identity, we obtain a celebrated identity of Jacobi,

$$(q; q)_\infty^3 = \sum_{k=0}^{\infty}(-1)^k (2k+1) q^{(k^2+k)/2}. \tag{13}$$

It might be noted that this can be written

$$(q; q)_\infty^3 = \sum_{k=-\infty}^{\infty} (4k+1) q^{2k^2+k}, \tag{14}$$

and this leads directly to a proof of the two–squares theorem!

Most, but not all, of this section is in Hardy and Wright, but it will all appear in my book "The power of q".

§3 Important special cases of Jacobi's triple product identity and Ramanujan's 5–dissection of Euler's product

We have (12)

$$(a^{-1}q, aq, q; q)_\infty = 1 + \sum_{k=1}^{\infty} (-1)^k (a^k + a^{k-1} + \cdots + a^{-k}) q^{(k^2+k)/2}.$$

If we set $a = \eta = \exp \dfrac{2\pi i}{5}$, and use the facts that

$$\eta + \eta^{-1} = -\beta, \quad \eta^2 + \eta^{-2} = -\alpha,$$

where $\alpha = \dfrac{1+\sqrt{5}}{2}$, $\beta = \dfrac{1-\sqrt{5}}{2}$ (Fibonacci notation, not Ramanujan's) and

$$\eta^k + \eta^{k-1} + \cdots + \eta^{-k} = \begin{cases} 1 & \text{if } k = 5l,\ l \geq 0, \\ \alpha & \text{if } k = 5l+1,\ l \geq 0, \\ 0 & \text{if } k = 5l+2,\ l \geq 0, \\ -\alpha & \text{if } k = -5l-2,\ l \leq -1, \\ -1 & \text{if } k = -5l-1,\ l \leq -1, \end{cases} \tag{15}$$

we find that

$$\prod_{k=1}^{\infty} (1 + \beta q^k + q^{2k})(1 - q^k) = (q^{10}, q^{15}, q^{25}; q^{25})_\infty - \alpha q (q^5, q^{20}, q^{25}; q^{25})_\infty. \tag{16}$$

If instead, we set $a = \eta^2$ (or simply change $\sqrt{5}$ to $-\sqrt{5}$), we find that

$$\prod_{k=1}^{\infty} (1 + \alpha q^k + q^{2k})(1 - q^k) = (q^{10}, q^{15}, q^{25}; q^{25})_\infty - \beta q (q^5, q^{20}, q^{25}; q^{25})_\infty. \tag{17}$$

Let

$$A = (q^{10}, q^{15}, q^{25}; q^{25})_\infty, \quad B = (q^5, q^{20}, q^{25}; q^{25})_\infty.$$

Then (16) and (17) can be written

$$A - \alpha q B = \prod_{k=1}^{\infty} (1 + \beta q^k + q^{2k})(1 - q^k) \tag{18}$$

and

$$A - \beta q B = \prod_{k=1}^{\infty} (1 + \alpha q^k + q^{2k})(1 - q^k). \tag{19}$$

In other words, both $A - \alpha q B$ and $A - \beta q B$ are infinite products.

We now multiply (18) and (19). On the left we obtain

$$(A - \alpha q B)(A - \beta q B) = A^2 - qAB - q^2 B^2. \tag{20}$$

On the right, using the marvellous fact that

$$(1 + \alpha x + x^2)(1 + \beta x + x^2) = 1 + x + x^2 + x^3 + x^4 = \frac{1 - x^5}{1 - x}, \tag{21}$$

we obtain

$$\begin{aligned} \prod_{k=1}^{\infty} (1 + \alpha q^k + q^{2k})(1 + \beta q^k + q^{2k})(1 - q^k)^2 &= \prod_{k=1}^{\infty} \frac{(1 - q^{5k})(1 - q^k)^2}{1 - q^k} \\ &= (q; q)_\infty (q^5; q^5)_\infty. \end{aligned} \tag{22}$$

So we have

$$A^2 - qAB - q^2 B^2 = (q; q)_\infty (q^5; q^5)_\infty. \tag{23}$$

If we divide by

$$qAB = q(q^5, q^{10}, q^{15}, q^{20}, q^{25}, q^{25}; q^{25})_\infty = q(q^5; q^5)_\infty (q^{25}; q^{25})_\infty \tag{24}$$

we obtain

$$\frac{A}{qB} - 1 - \frac{qB}{A} = \frac{(q; q)_\infty}{q(q^{25}, q^{25})_\infty}. \tag{25}$$

We can write this

$$(q; q)_\infty = (q^{25}; q^{25})_\infty \left\{ \left(\begin{matrix} q^{10}, q^{15} \\ q^5, q^{20} \end{matrix} ; q^{25} \right)_\infty - q - q^2 \left(\begin{matrix} q^5, q^{20} \\ q^{10}, q^{15} \end{matrix} ; q^{25} \right)_\infty \right\}. \tag{26}$$

This is Ramanujan's 5–dissection of Euler's product.

To see what has been achieved, observe what follows. We know that Euler's product,

$$(q; q)_\infty = \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2 - n)/2} = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + + - - \;\cdots\; .$$

Notice that the powers are congruent only to 0, 1 or 2 modulo 5. We can bracket the terms as follows,

$$(q;q)_\infty = (1+q^5-q^{15}+\ \cdots\ ) - q(1-q^{25}-\ \cdots\ ) - q^2(1-q^5+\ \cdots\ ),$$

and then it turns out that the series within each pair of brackets is a product,

$$\begin{aligned}(q;q)_\infty &= \left(\begin{matrix} q^{10},q^{15},q^{25} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty - q(q^{25};q^{25})_\infty - q^2\left(\begin{matrix} q^5,q^{20},q^{25} \\ q^{10},q^{15} \end{matrix};q^{25}\right)_\infty \\ &= (q^{25};q^{25})_\infty \left\{ \left(\begin{matrix} q^{10},q^{15} \\ q^5,q^{20} \end{matrix};q^{25}\right)_\infty - q - q^2\left(\begin{matrix} q^5,q^{20} \\ q^{10},q^{20} \end{matrix};q^{25}\right)_\infty \right\}, \qquad (27)\end{aligned}$$

as Ramanujan states.

The pair of identities (18) and (19), which when multiplied together yield (27), are said to constitute a factorisation of (27). Hence the title of this talk.

§4 Ramanujan's "most beautiful identity"

We start by writing

$$E(q) = (q;q)_\infty = \prod_{k=1}^\infty (1-q^k),$$

and then

$$\sum_{n=0}^\infty p(n)q^n = \frac{1}{E(q)} = \frac{E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)}{E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)}. \qquad (28)$$

The denominator is

$$
\begin{aligned}
&E(q)E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)\\
&= \prod_{k=1}^{\infty}(1-q^k)(1-\eta^k q^k)(1-\eta^{2k}q^k)(1-\eta^{3k}q^k)(1-\eta^{4k}q^k)\\
&= \prod_{k\equiv 0 \pmod 5}(1-q^k)^5 \cdot \prod_{k\not\equiv 0 \pmod 5}(1-q^{5k})\\
&= \prod_{k=1}^{\infty}(1-q^{5k})^5 \cdot \prod_{k=1}^{\infty}(1-q^{5k}) \Big/ \prod_{k\equiv 0 \pmod 5}(1-q^{5k})\\
&= \prod_{k=1}^{\infty}(1-q^{5k})^6 \Big/ \prod_{k=1}^{\infty}(1-q^{25k})\\
&= \frac{(q^5;q^5)_\infty^6}{(q^{25};q^{25})_\infty},
\end{aligned}
\tag{29}
$$

while the numerator is

$$
\begin{aligned}
&E(\eta q)E(\eta^2 q)E(\eta^3 q)E(\eta^4 q)\\
&= (q^{25};q^{25})_\infty^4 \left\{R(q^5)^{-1} - \eta q - \eta^2 q^2 R(q^5)\right\}\left\{R(q^5)^{-1} - \eta^2 q - \eta^4 q^2 R(q^5)\right\}\\
&\qquad \times \left\{R(q^5)^{-1} - \eta^3 q - \eta^6 q^2 R(q^5)\right\}\left\{R(q^5)^{-1} - \eta^4 q - \eta^8 q^2 R(q^5)\right\}\\
&= (q^{25};q^{25})_\infty^4 \left\{R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2 R(q^5)^{-2} + 3q^3 R(q^5)^{-1} + 5q^4\right.\\
&\qquad \left. -3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4\right\}.
\end{aligned}
\tag{30}
$$

Thus from (28), (29) and (30) we find

$$
\begin{aligned}
\sum_{n=0}^{\infty} p(n)q^n = \frac{(q^{25};q^{25})_\infty^5}{(q^5;q^5)_\infty^6}&\left\{R(q^5)^{-4} + qR(q^5)^{-3} + 2q^2 R(q^5)^{-2} + 3q^3 R(q^5)^{-1}\right.\\
&\left. +5q^4 - 3q^5 R(q^5) + 2q^6 R(q^5)^2 - q^7 R(q^5)^3 + q^8 R(q^5)^4\right\}
\end{aligned}
\tag{31}
$$

If we extract those terms in which the power of q is 4 (mod 5), divide by q^4 and replace q^5 by q, we obtain

$$
\sum_{n=0}^{\infty} p(5n+4)q^n = 5\frac{(q^5;q^5)_\infty^5}{(q;q)_\infty^6},
\tag{32}
$$

Ramanujan's "most beautiful identity".

§5 Further factorisations

Let

$$A=(q^3,q^4,q^6,q^7,q^{10},q^{10};q^{10})_\infty,\quad B=(q,q^2,q^8.q^9,q^{10},q^{10};q^{10})_\infty$$

and

$$k=\frac{qB}{A}=\left(\begin{matrix}q,q^2,q^8,q^9\\ q^3,q^4,q^6,q^7\end{matrix};q^{10}\right)_\infty,$$

as in (4).

We can prove the following three pairs of identities.

$$A+qB=(-q,q^2,q^3,-q^4,q^5,q^5;q^5)_\infty, \tag{33}$$

$$A-qB=(q,-q^2,-q^3,q^4,q^5,q^5;q^5)_\infty, \tag{34}$$

$$A+\alpha qB=\prod_{k=1}^{\infty}(1+\alpha q^{2k-1}+q^{4k-2})(1+\beta q^{2k}+q^{4k})(1-q^{2k})^2, \tag{35}$$

$$A+\beta qB=\prod_{k=1}^{\infty}(1+\beta q^{2k-1}+q^{4k-2})(1+\alpha q^{2k}+q^{4k})(1-q^{2k})^2, \tag{36}$$

and

$$A-(2-\sqrt{5})qB=\prod_{k=1}^{\infty}(1+\alpha q^{k}+q^{2k})(1-\beta q^{k}+q^{2k})(1-q^{k})^2, \tag{37}$$

$$A-(2+\sqrt{5})qB=\prod_{k=1}^{\infty}(1+\beta q^{k}+q^{2k})(1-\alpha q^{k}+q^{2k})(1-q^{k})^2. \tag{38}$$

If we multiply the first pair, we obtain

$$A^2-q^2B^2=(q^2,q^4,q^6,q^8;q^{10})_\infty(q^5;q^5)_\infty^4=\frac{(q^2;q^2)_\infty(q^5;q^5)_\infty^4}{(q^{10};q^{10})_\infty}. \tag{39}$$

If we divide (39) by

$$qAB=q\frac{(q;q)_\infty(q^{10};q^{10})_\infty^4}{(q^5;q^5)_\infty} \tag{40}$$

we obtain

$$\frac{A}{qB} - \frac{qB}{A} = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5},$$

or,

$$\frac{1}{k} - k = \frac{(q^2;q^2)_\infty (q^5;q^5)_\infty^5}{q(q;q)_\infty (q^{10};q^{10})_\infty^5},$$

which is (5). Thus the pair (33), (34) constitutes a factorisation of (5).

If we multiply the second pair, we find

$$A^2 - qAB - q^2B^2 = \prod_{k=1}^{\infty} \frac{(1-q^{10k-5})}{(1-q^{2k-1})} \frac{(1-q^{10k})}{(1-q^{2k})} (1-q^{2k})^4 = \frac{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty}{(q;q)_\infty},$$

and if we divide by qAB,

$$\frac{A}{qB} - 1 - \frac{qB}{A} = \frac{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty^2}{q(q;q)_\infty^2 (q^{10};q^{10})_\infty^4},$$

or,

$$\frac{1}{k} - 1 - k = \frac{(q^2;q^2)_\infty^4 (q^5;q^5)_\infty^2}{q(q;q)_\infty^2 (q^{10};q^{10})_\infty^4},$$

which is (6). So the pair (35), (36) constitute a factorisation of (6).

If we multiply (37) and (38), we obtain

$$A^2 - 4qAB - q^2B^2 = \prod_{k=1}^{\infty} \frac{(1-q^{5k})}{(1-q^{k})} \frac{(1+q^{5k})}{(1+q^{k})} (1-q^{k})^4 = \frac{(q;q)_\infty^4 (q^{10};q^{10})_\infty}{(q^2;q^2)_\infty},$$

and if we divide by qAB, we obtain

$$\frac{A}{qB} - 4 - \frac{qB}{A} = \frac{(q;q)_\infty^3 (q^5;q^5)_\infty}{q(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3},$$

or,

$$\frac{1}{k} - 4 - k = \frac{(q;q)_\infty^3 (q^5;q^5)_\infty}{q(q^2;q^2)_\infty (q^{10};q^{10})_\infty^3},$$

which is (7). So the pair (37), (38) constitute a factorisation of (7).

§6 Some proofs

First we prove (33).

The right side of (33) is, by Jacobi's triple product identity,

$$(-q,q^2,q^3,-q^4,q^5,q^5;q^5)_\infty = \sum_{r,s=-\infty}^{\infty}(-1)^s q^{(5r^2-3r+5s^2-s)/2}.$$

We split this sum in two, according to whether $r\equiv s \pmod 2$ or $r\not\equiv s \pmod 2$. In the first case, let $r+s=2t,\ r-s=2u,\ r=t+u,\ s=t-u$, and in the second case, let $r+s=2t+1,\ r-s=2u+1,\ r=t+u+1,\ s=t-u$, and we find

$$\begin{aligned}
&(-q,q^2,q^3,-q^4,q^5,q^5;q^5)_\infty \\
&= \sum_{t,u=-\infty}^{\infty}(-1)^{t-u}q^{(5(t+u)^2-3(t+u)+5(t-u)^2-(t-u))/2} \\
&\quad + \sum_{t,u=-\infty}^{\infty}(-1)^{t-u}q^{(5(t+u+1)^2-3(t+u+1)+5(t-u)^2-(t-u))/2} \\
&= \sum_{t,u=-\infty}^{\infty}(-1)^{t+u}q^{5t^2-2t+5u^2-u} + q\sum_{t,u=-\infty}^{\infty}(-1)^{t+u}q^{5t^2+3t+5u^2+4u} \\
&= (q^3,q^4,q^6,q^7,q^{10},q^{10};q^{10})_\infty + q(q,q^2,q^8,q^9,q^{10},q^{10};q^{10})_\infty,
\end{aligned}$$

which is the left side of (33). The proof of (34) is similar, so omitted.

The proof of (35) is a little harder, and we shall omit some details.

We have Jacobi's triple product identity

$$(-a^{-1}q,-aq,q^2;q^2)_\infty = 1+\sum_{k=1}^{\infty}(a^k+a^{-k})q^{k^2}. \tag{41}$$

If we set $a=\exp\{\frac{i\pi}{5}\}$, we obtain

$$\begin{aligned}
&\prod_{k=1}^{\infty}(1+\alpha q^{2k-1}+q^{4k-2})(1-q^{2k}) \\
&= (q^{25},q^{25},q^{50};q^{50})_\infty + \alpha q(q^{15},q^{35},q^{50};q^{50})_\infty - \beta q^4(q^5,q^{45},q^{50};q^{50})_\infty.
\end{aligned} \tag{42}$$

If we put q^2 for q in (16), we obtain

$$\prod_{k=1}^{\infty}(1+\beta q^{2k}+q^{4k})(1-q^{2k}) = (q^{20},q^{30},q^{50};q^{50})_\infty - \alpha q^2(q^{10},q^{40},q^{50};q^{50})_\infty. \tag{43}$$

If we multiply (42) by (43), and use the facts that $\alpha^2=\alpha+1$ and $\beta=1-\alpha$, we find that

$$
\begin{aligned}
&\prod_{k=1}^{\infty}(1+\alpha q^{2k-1}+q^{4k-2})(1+\beta q^{2k}+q^{4k})(1-q^{2k})^2\\
&=\big\{(q^{20},q^{25},q^{25},q^{30},q^{50},q^{50};q^{50})_\infty\\
&\quad -q^3(q^{10},q^{15},q^{35},q^{40},q^{50},q^{50};q^{50})_\infty\\
&\qquad -q^4(q^{5},q^{20},q^{30},q^{45},q^{50},q^{50};q^{50})_\infty\\
&\qquad\quad -q^6(q^{5},q^{10},q^{40},q^{45},q^{50},q^{50};q^{50})_\infty\big\}\\
&+\alpha q\,\big\{(q^{15},q^{25},q^{25},q^{35},q^{50},q^{50};q^{50})_\infty\\
&\quad -q(q^{10},q^{25},q^{25},q^{40},q^{50},q^{50};q^{50})_\infty\\
&\qquad -q^2(q^{10},q^{15},q^{35},q^{40},q^{50},q^{50};q^{50})_\infty\\
&\qquad\quad +q^3(q^{5},q^{20},q^{30},q^{45},q^{50},q^{50};q^{50})_\infty\big\}\,. \qquad (44)
\end{aligned}
$$

Now believe me when I tell you that the first pair of brackets on the right of (44) contains the 5–dissection of $(q^3,q^4,q^6,q^7,q^{10},q^{10};q^{10})_\infty$, and the second pair of brackets contains the 5–dissection of $(q,q^2,q^8,q^9,q^{10},q^{10};q^{10})_\infty$.

(My only reason for omitting this part of the proof is that it is too time–consuming.)

So (44) becomes

$$
\begin{aligned}
&\prod_{k=1}^{\infty}(1+\alpha q^{2k-1}+q^{4k-2})(1+\beta q^{2k}+q^{4k})(1-q^{2k})^2\\
&=(q^3,q^4,q^6,q^7,q^{10},q^{10};q^{10})_\infty+\alpha q(q,q^2,q^8,q^9,q^{10},q^{10};q^{10})_\infty,
\end{aligned}
$$

which is (35). We can obtain (36) from (35) by replacing $\sqrt{5}$ by $-\sqrt{5}$.

The proof of (37) (and (38)) is a little harder again, so I omit that, as well.

§7 The Rogers–Ramanujan continued fraction

The result that

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cdots}}}}} = \left(\begin{matrix} q, q^4 \\ q^2, q^3 \end{matrix}; q^5\right)_\infty,$$

known as the Rogers–Ramanujan continued fraction, is the special case $a = 1$ of

$$\cfrac{1}{1+\cfrac{aq}{1+\cfrac{aq^2}{1+\cfrac{aq^3}{1+\cfrac{aq^4}{1+\cdots}}}}} = \frac{\displaystyle\sum_{k=0}^{\infty}(-1)^k a^{2k} q^{(5k^2+3k)/2}(1-aq^{2k+1})\frac{(aq;q)_k}{(q;q)_k}}{\displaystyle\sum_{k=0}^{\infty}(-1)^k a^{2k} q^{(5k^2+k)/2}(1-a^2q^{4k+2})\frac{(aq;q)_k}{(q;q)_k}}.$$

One can prove this result as follows. Let $G(a)$ denote the denominator of the right side. It can quite easily be shown following Ramanujan that

$$\frac{G(a)}{1-aq} = G(aq) + aq(1-aq^2)G(aq^2).$$

(see my book for details).
This can be written

$$\frac{G(a)}{(1-aq)G(aq)} = 1 + aq \cdot \frac{(1-aq^2)G(aq^2)}{G(aq)},$$

or,

$$\frac{(1-aq)G(aq)}{G(a)} = \cfrac{1}{1+aq\cdot\cfrac{(1-aq^2)G(aq^2)}{G(aq)}}.$$

The result now follows by iteration.

That is where I will finish.

Thank you!