MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 9, July 24, 2017

1. What is the least positive integer n such that $90 \times n$ is a cube?

2. Show that any straight line passing through the centre of a parallelogram (i.e. the intersection of the diagonals) divides the parallelogram into two equal areas.

3. A mathematics test has 5 questions on each of which people can score 0,1,2 or 3 marks. How many ways can a student receive a total of 12 marks for the test?

4. Use the fact that $2xy = (x + y)^2 - x^2 - y^2$ to show that

$$2(b - c)(c - a) + 2(c - a)(a - b) + 2(a - b)(b - c) \leq 0$$

   for all real numbers a, b, c.

5. Take any triangle ABC and show how to construct an equilateral triangle inside ABC whose vertices touch the sides of ABC. (Hint: Start by constructing an equilateral triangle outside ABC with AB as one of its sides.)

6. Imagine that we have a finite set A of integer numbers, that is, a collection of integers without repetition. Consider the set $A + A$ of all possible sums of two numbers in A:
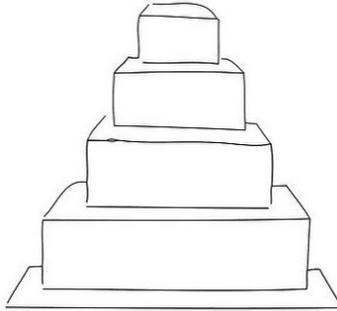
$$A + A = \{n : \ n = a + a' \text{ for some numbers } a, a' \text{ in } A\}.$$

   We denote how many numbers there are in the set A by $|A|$.

   (a) Show that $|A + A| \geq 2|A| - 1$.
   (b) Show that if $|A + A| = 2|A| - 1$, then A is an arithmetic progression.

Senior Questions

1. Imagine that you have a square based cake, like the one in the picture.

   (a) How would you cut it into 5 pieces of equal volume? How about 7 pieces?

   (b) How about n pieces of equal volume?

2. Show that $\log_2 3$ is not a rational number.