



**MATHEMATICS ENRICHMENT CLUB.**

**Problem Sheet 13, August 21, 2017**

1. Given that  $x$  and  $y$  are integers, find all solutions to

$$3x^2 - 8xy + 4y^2 = -12$$

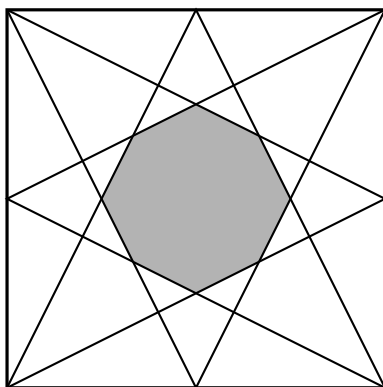
2. Write the quartic  $x^4 + 4$  as the product of two quadratics. What about  $x^4 + 1$ ?

3. Find all positive integers  $x, y$  and  $z$  such that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{5}{8}.$$

(Hint: Suppose  $x \leq y \leq z$  and hence find the possible values of  $x$ .)

4. An octagon is created by joining the vertices and midpoints of the sides of a unit square as shown below.



Calculate the area of the octagon.

5. In how many ways is it possible to write 1000 as a sum of consecutive odd integers?
6. Let  $n$  be an integer greater than 1. The tau-function,  $\tau(n)$  is defined as the number of divisors of  $n$  (including  $n$  itself). For example, the divisors of 6 are 1, 2, 3 and 6, so

$$\tau(6) = 4.$$

- (a) Evaluate  $\tau(7)$ ,  $\tau(10)$  and  $\tau(25)$ .
- (b) What can you say about a number  $m$  if  $\tau(m) = 2$ ? What if  $\tau(m)$  is odd?
- (c) Determine a formula for  $\tau(n)$ .

### Senior Questions

1. Find the sum

$$S = \frac{1}{1 \times 4} + \frac{1}{4 \times 7} + \dots + \frac{1}{(3n-2)(3n+1)}$$

2. Let  $I = \int \sec \theta d\theta$ .

In this question, we will evaluate  $I$  in two different ways.

- (a) **METHOD I:** Show that

$$\sec \theta = \frac{\cos \theta}{1 - \sin^2 \theta}.$$

Hence evaluate  $I$ .

- (b) **METHOD II:** Show that if  $f(\theta) = \sec \theta + \tan \theta$ , then

$$\frac{f'(\theta)}{f(\theta)} = \frac{\sec \theta (\sec \theta + \tan \theta)}{(\sec \theta + \tan \theta)}.$$

Hence evaluate  $I$ .

- (c) Reconcile the results of Method I and Method II.

3. Let  $n$  be an integer greater than 1. The sigma-function,  $\sigma(n)$  is defined as the sum of the divisors of  $n$  (including  $n$  itself). For example, the divisors of 6 are 1, 2, 3 and 6, so

$$\sigma(6) = 1 + 2 + 3 + 6 = 12.$$

Find a formula for  $\sigma(n)$ .