



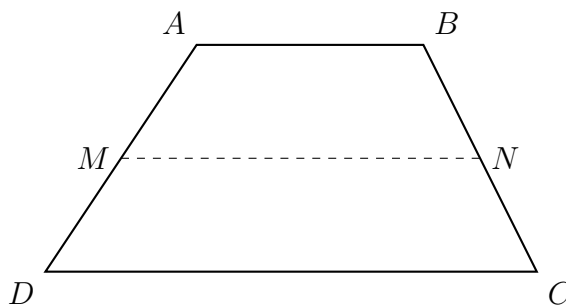
MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 14, August 28, 2017

1. The product of the ages in years of two adults is 770. What is the sum of their ages?
2. Calculate

$$P = \left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{2017}\right)$$

3. (a) Prove that $a + b \geq 2\sqrt{ab}$ for any positive real numbers a and b . When does equality hold?
(b) Deduce that, for x, y and z positive, $(x + y)(x + z)(y + z) \geq 8xyz$.
4. Let $ABCD$ be a trapezium with $AB \parallel CD$. Let M and N be the midpoints of AD and BC respectively. Show that $MN = \frac{1}{2}(AB + CD)$. (MN is called the midline of the trapezium.)



5. Construct a trapezium given one of its angles, both diagonals and the midline.
6. If $x_1 = 3$, the recurrence $x_{n+1} = x_n^2 - 10x_n$ gives the sequence $3, -21, 651, 417291, \dots$ and the numbers increase without bound. Find all the values of x_1 so that the sequence does NOT increase without bound.

Senior Questions

1. Suppose that m and n are positive real numbers. Use trigonometry to find the maximum value of

$$\frac{m+n}{\sqrt{m^2+n^2}}.$$

2. Let $I = \int_0^1 \frac{x^4(x-1)^4}{x^2+1} dx$.

(a) Evaluate I .

(b) Hence deduce that $\pi < \frac{22}{7}$.

3. The hyperbolic sine and cosine functions ($\sinh x$ and $\cosh x$) are defined as

$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x}) \\ \cosh x &= \frac{1}{2}(e^x + e^{-x})\end{aligned}$$

(a) Prove the following identity.

$$\cosh^2 x - \sinh^2 x = 1.$$

(b) If $\sinh x = \tan \theta$, express $\cosh x$ in terms of θ .