



MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 17, September 18, 2017

1. Determine the last digit of 2017^{2017} .
2. Find all positive integer solutions to

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{6}$$

3. The perimeter of a rectangle is 20 cm. What is the least value of the diagonal?
4. (a) Verify that

$$\begin{aligned}x^{15} - 1 &= (x^3 - 1)(x^{12} + x^9 + x^6 + x^3 + 1) \\ &= (x^5 - 1)(x^{10} + x^5 + 1)\end{aligned}$$

- (b) Hence factor $2^{15} - 1$ as a product of prime factors.
- (c) Can you factorise $2^{15} + 1$ as a product of primes?
5. Draw a regular 5-pointed star. Find the sum of the angles at the 5 points of the star. What happens in an n -pointed star?
6. Describe how to construct a triangle ABC if we are given the lengths of the sides AB and AC and the length of the median AM drawn from A .

Senior Questions

1. Take any (fixed) positive integer m and let $P_n = (m+1)(m+2)\dots(m+n)$. Prove that $n!$ is always a factor of P_n . (For example, $6! = 720$ is a factor of $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 = 1\,235\,520$.)
2. The hyperbolic sine and cosine functions ($\sinh x$ and $\cosh x$) are defined as

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}).$$

- (a) Show that $\frac{d}{dx} \sinh x = \cosh x$.
- (b) Find a formula for the inverse of $\sinh x$.
- (c) Hence use the substitution $x = \sinh u$ to deduce that

$$\int \frac{dx}{\sqrt{x^2 + 1}} = \ln(x + \sqrt{x^2 + 1}) + C.$$

3. Find the stationary point(s) of $f(x) = x^{2x}$, and prove that the graph of f is always concave up.