

**MATHEMATICS ENRICHMENT CLUB.<sup>1</sup>**

**Problem Sheet 3, May 14, 2012**

1. The perimeter of a base of a rectangular brick with integer sides is 18 cm, whilst its volume is  $42 \text{ cm}^3$ . What is its height?

2. Calculate

$$\left(1 - \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{2008}\right).$$

3. Find the smallest positive integer whose square ends in (a) 09 and (b) 9009.

4. Show that if  $a, b$  are positive numbers such that  $ab \leq 1$  then

$$\frac{a}{b+1} + \frac{b}{a+1} + (1-a)(1-b) \leq 1.$$

5. Suppose we have the numbers  $x_0 = 0, x_1 = 1$  and  $x_{n+1} = x_n + 2x_{n-1}$  for  $n \geq 2$ .

a. Write down the numbers  $x_n$  for  $n = 2, 3, 4, 5, 6$ .

b. Show that there is no  $n$  for which  $x_n = 1999$ . (Hint: Use modulo 8 arithmetic).

c. Show that  $x_n = \frac{2^n - (-1)^n}{3}$  satisfies the equation.

6. In  $\triangle ABC$ , extend the sides  $AB$  and  $AC$  and draw a circle outside the triangle which touches  $BC$  and these two produced sides. This circle is called the *escribed circle* of the triangle.

a. Show that  $r_1 = \frac{A}{s-a}$ , where  $r_1$  is the radius of the escribed circle,  $A$  is the area of  $ABC$ ,  $a$  is the length of  $BC$  and  $s$  is half the perimeter of  $ABC$ .

b. Show that  $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}$ , where  $r_2, r_3$  are the radii of the other two escribed circles and  $r$  is the radius of the incircle (recall last weeks result.)

7.  $ABCD$  is a parallelogram,  $Q$  a point inside it. Prove that the sum of the areas of  $AQB$  and  $CQD$  is half the area of  $ABCD$ .

<sup>1</sup>Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres, Macquarie Uni.

**Senior Questions.**

1. Find  $\int_0^1 \frac{1}{1+t+t^2}$ .

2. Find the limit  $\lim_{n \rightarrow \infty} \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{n^3}$ .

3. A hand of eight cards is dealt from a standard pack. How many hands contain exactly three cards of the same value and the remaining cards from the remaining suit?