UNSW THE UNIVERSITY OF NEW SOUTH WALES

Never Stand Still | Faculty of Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 12, August 14, 2012

1. The number 2012 uses just three digits. How many years since 1000 AD have used just three digits?

2. Calculate the product $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\ldots\left(1-\frac{1}{100}\right)$.

3. (a) Express $\frac{1}{3-\sqrt{7}}$ with a rational denominator.
   (b) Do the same for $\frac{1}{3-\sqrt[3]{7}}$.

4. Suppose that a, b, c are positive **odd** integers such that $ab + bc + ca = 215$, where $a \le b \le c$.
   (a) Show that $a^2 + ab + b^2 \le 215$ and hence find the largest possible value of b.
   (b) Hence find all the possible triples a, b, c.

5. Given an ordered triple of non-zero numbers (a, b, c), we produce a new triple (ab, bc, ca). For example, $(1, 2, 3) \to (2, 6, 3)$. Suppose we repeat this process a number of times. Show that we generally never return to where we start, but that if we do, then it will happen in at most 6 steps. Can you find triples which return to themselves after 1,2,3,4,5, or 6 steps?

6. Let ABC be a triangle with three medians intersecting at S. Let L, M be the midpoints of AC, AB respectively.
   (a) Prove that the triangles LSC and MSB have equal areas.
   (b) Given that LSC has area 100cm^2, find the area of ABC.

7. Let $ABCD$ be a tetrahedron with skew edges AB, CD. (Two edges are *skew* if they don't lie in the same plane.)

   Name the edge which is skew to BC and the one skew to BD.

   The line that joins the midpoints of a pair of skew edges is called an *edge-bisector*. Show that the three edge bisectors of a tetrahedron intersect at a single point which is the midpoint of each edge bisector.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.