MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 5, June 3, 2014[1]

1. In the equation
$$29 + 38 + 10 + 4 + 5 + 6 + 7 = 99,$$
the left hand side contains each digit exactly once. Either find a similar expression using all the digits 0-9 and only + signs to obtain 100 or prove that it isn't possible.

2. Show that the fraction
$$\frac{21n + 4}{14n + 3}$$
cannot be simplified further for any positive integer n.

3. A point P lies inside a triangle ABC. Three lines are drawn through P parallel to the sides of ABC dividing the triangle into 6 regions, 3 of which are triangles. If the area of these smaller triangles are 12, 27 and 75 square centimetres, find the area of ABC.

4. A bakery sells donuts in packs of 5, 9 or 13. What is the largest number of donuts that cannot be bought exactly?

5. Let a_n be the Fibonacci sequence, i.e.
$$a_1 = 1, \quad , a_2 = 1, \quad a_n = a_{n-1} + a_{n-2} \text{ for } n \geq 3.$$
Prove the following:
   (a) $a_n^2 - a_{n-1}a_{n+1} = (-1)^{n-1}$
   (b) a_n is even if and only if n is a multiple of 3.
   (c) $a_n = a_{r+1}a_{n-r} + a_r a_{n-r-1}$ for $1 \leq r \leq n-2$
   (d) a_k is a factor of a_n if and only if k is a factor of n.

[1] Some problems from UNSW's publication *Parabola*

Senior Questions A $n \times n$ square matrix is table of numbers that has n rows and n columns. We can write the entry in the ith row and jth column of a matrix A as $[A]_{ij}$. So

$$A = \begin{pmatrix} [A]_{11} & [A]_{12} & \cdots & & [A]_{1n} \\ [A]_{21} & [A]_{22} & [A]_{23} & \cdots & [A]_{2n} \\ [A]_{31} & & \ddots & & \\ \vdots & & & & \\ [A]_{n1} & \cdots & & & [A]_{nn} \end{pmatrix}$$

Just like numbers, square matrices of the same size can be added and multiplied together. Two matrices are equal if all of their entries are equal.

1. To add square matrices of the same size together, we simply add their corresponding entries. That is $[A+B]_{ij} = [A]_{ij} + [B]_{ij}$. Prove that square matrix addition is commutative and associative, i.e.

   (a) $A + B = B + A$ (commutative) and

   (b) $A + (B + C) = (A + B) + C$ (associative).

2. To multiply square matrices together we follow the rule

$$[AB]_{ij} = \sum_{k=1}^{n} [A]_{ik}[B]_{kj}.$$

   Show that matrix multiplication

   (a) is associative, i.e. $A(BC) = (AB)C$, but

   (b) is not commutative, i.e. $AB \neq BA$ for all $n \times n$ square matrices A and B.

3. With real numbers we have a special number, 1, which if you multiply any number, $x \neq 0$, to it you get the same number x back, i.e. $x1 = 1x = x$. There is a similar matrix I which has the rule that for any matrix A such that not every entry is 0 (there's at least one entry $[A]_{ij} \neq 0$) we have $AI = IA = A$. Find the matrix I.