MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 7, June 17, Brazil 2014[1]

1. Let $x = \sqrt[3]{10} + \sqrt[3]{6}$. Show that $x^3 - 3x\sqrt[3]{60} = 16$ and deduce (without a calculator) that $x < 4$.

2. A non-empty, finite, connected graph on a sphere is a finite collection of points, called nodes or vertices, and lines joining them, called edges, drawn on to a sphere. Every vertex is connected to at least one other by an edge, and every edge connects exactly two vertices. The edges divide the sphere into a number of regions called faces.

   (a) Consider a non-empty, finite, connected graph on a sphere with v vertices, e edges and f faces. We can delete a vertex by merging it with another vertex on the other end of an edge connecting the two. What happens to v, e and f in the case that the deleted vertex is: a) part of a triangle? b) not part of a triangle?

   (b) For any non-empty, connected, finite graph on a sphere, show that $v - e + f = 2$.

   (c) A standard soccer ball is a non-empty, finite, connected graph on a sphere which satisfies

   - Each face is either a pentagon or a hexagon,
   - the sides of each pentagon meet only hexagons,
   - the sides of each hexagon alternately meet pentagons and hexagons, and
   - precisely three edges meet at every vertex.

   Find the number of pentagons and hexagons on a standard soccer ball.

3. In the group stage of the world cup 4 teams make up a group. Each team plays the other 3 in their group and are awarded 3 points for a win, 1 for a draw and zero for a loss. After all 6 matches are played the top two advance. Australia is in group B with Spain, Chile and the Netherlands. As of writing this, Chile has defeated Australia and the Netherlands have defeated Spain. The outcomes of the remaining matches occur with the following probabilities:

[1] Some problems from UNSW's publication *Parabola*, the Mathematics Enrichment club is in no way affiliated with FIFA or the greatest sport in the world.

Team 1	vs	**Team 2**	Team 1 wins	Draw	Team 2 wins
Australia	vs	**Netherlands**	$\frac{1}{13}$	$\frac{8}{65}$	$\frac{4}{5}$
Spain	vs	**Chile**	$\frac{4}{7}$	$\frac{8}{35}$	$\frac{1}{5}$
Australia	vs	**Spain**	$\frac{1}{15}$	$\frac{1}{10}$	$\frac{5}{6}$
Chile	vs	**Netherlands**	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{5}{12}$

What are Australia's chances of advancing? (We will use my own amendments to the rules in which any ties in points after all games are played are resolved by a coin flip.)

4. How can a regular hexagon be dissected by straight line cuts into five pieces which can then be re-assembled to make a square?

5. Let $ABCD$ be a rectangle, and K, L be the midpoints of AB and CD respectively. Suppose AC and KD meet at X. Find the ratio of the area of ΔKXA to the rectangle $ABCD$.

Senior Questions

1. Let $f : [a, b] \to \mathbb{R}$ be a continuous function for $a \leq x \leq b$ with $f(a) = c$ and $f(b) = d$. Show that for every real number, z, between c and d, there's a y with $f(y) = z$.

2. Let $f : [a, b] \to [a, b]$ be a continuous function fo $a \leq x \leq b$ such that $a \leq f(x) \leq b$. Show that there's a *fixed point* of f, that is, there's a p, $a \leq p \leq b$ with $f(p) = p$.

3. Let $g : [a, b] \to \mathbb{R}$ be continuous for $a \leq x \leq b$ and differentiable for $a < x < b$ with $g(a) = g(b)$. Show that there's a c, $a < c < b$ such that $g'(c) = 0$.

4. Let g be as above. Show that there's a c, $a < c < b$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$