MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 8, June 24, 2014[1]

1. Website A demands its users have a password that is 6–8 characters long, contain only letters or numbers and must contain at least three letters and three numbers but is case insensitive, while website B demands only that its users have passwords that are no more than 6 characters long, use only letters or numbers and is case sensitive. Which website has more possible passwords?

2. A thousand coins lie in one long row, all facing tails up. We first walk along the line and flip over every coin. Then we start again, flipping over every second coin, then again flipping over every third coin, then again flipping every forth and so on. After doing this a thousand times, so that the last time we do it, we flip only the last (one thousandth) coin, how many coins are facing heads up?

3. Three primes p, q, r satisfy the equation
$$p^2q^2 + r^2 = 2pqr + pq + r.$$
What is $p + q + r$?

4. Each side of a triangle of area 1 is divided into three equal parts. The six points of division are the vertices of two triangles whose intersection is a hexagon. Find the area of the hexagon.

5. We write $a + \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}}$ as $a + n$, just as we write $\underbrace{a + a + a + \cdots + a}_{n \text{ times}}$ as $a \times n$ and $\underbrace{a \times a \times \cdots \times a}_{n \text{ times}}$ as a^n. Let's keep going and write $\underbrace{a^{a^{\cdot^{\cdot^{\cdot^{a}}}}}}_{n \text{ times}}$ as ${}^n a$ (this is called *tetration*). Which is larger, ${}^9 8$ or ${}^8 9$?

6. Consider all subsets of 8 elements of the set $\{1, 2, 3, \ldots, 17\}$. From each subset select the smallest number. Show that the arithmetic mean of the 24 310 numbers selected is equal to 2.

[1]Some problems from UNSW's publication *Parabola*

Senior Questions

1. A right cone is a cone whose apex lies exactly above the centre of the base. The volume of a right cone is $\frac{1}{3}bh$ where b is the area of the base, and h the perpendicular height of the cone. An oblique cone is a cone whose apex does not lie exactly above the centre of the base. Show that the volume of an oblique cone has the same formula as a right cone.

2. A glass in the shape of an inverted, truncated, right cone (i.e. frustrum) contains some water. The top of the glass has radius R, the bottom has radius r, $r < R$, and the glass is h tall. When tilted, the surface of the water can be made to reach from the tip of base to the edge of the top. Find the proportion of the volume of the glass that the water takes up.