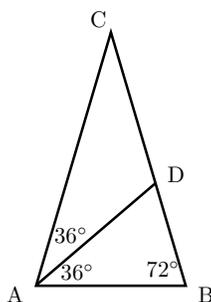




MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 5, May 26, 2015¹

1. An arithmetic sequence formed from 11 terms, and the sum of all its terms equals to 220. Find the middle term in that sequence.
2. (a) At a birthday party, the number of guess Bernard invited is equal to his age. During the party, each person shakes hands with another person exactly once. There are a total of 253 handshakes during the party, how old is Bernard?
(b) In the middle of the party, Bernard arranges his guess and himself to be seated at two round tables. If Bernard would like the number of people sitting in each table to be as even as possible, in how many ways can he do this (two arrangements are count as the same if they can be obtained by rotating the tables)?
3. Prove that seven is the only prime number that is one less than a perfect cube.



4. Let $\triangle ABC$ be a isosceles triangle with base $\angle BAC = \angle CBA = 72^\circ$, D is the point of intersection between the bisector of $\angle BAC$ and the line CB ; see above
 - (a) Let the length of CD and DB be a and b respectively. Find $\frac{a}{b}$.
 - (b) Express $\cos(36^\circ)$ in terms of a and b , and thus using part (a), find an exact expression for $\cos(36^\circ)$.
5. The number 4 has an odd number of odd positive divisors, namely 1, and an even number of even positive divisors, namely 2 and 4. Is there a number with an odd number of even positive divisors and an even number of odd positive divisors?

¹Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*. Q4 is by Adam. S

6. a and b are positive integers. Of the following statements, three are true, one is false.

- (i) $a + 1$ is divisible by b .
- (ii) $a = 2b + 5$.
- (iii) $a + b$ is divisible by 3.
- (iv) $a + 7b$ is a prime number.

Determine the possible values of a and b .

Senior Questions

1. (a) Let n be a positive integer. Prove that

$$a^n - 1 = (a - 1)(a^n + a^{n-1} + \dots + a + 1),$$

for all real numbers a .

(b) Given that $n \geq 2$ and an $a^n - 1$ is prime. Prove that $a = 2$, and n must be prime.

2. A, B and C are to fight a 3-corner pistol duel by firing at their choice of target in turns. A fires first, then B then C , then A, B, C etc, with a person dropping out when hit, until only one person remains.

When taking careful aim, A has a 30% of hitting this target, C has 50% chance and B never misses. What should A 's strategy be?

Repeat the problem when C has only a 40% chance of hitting his target.

3. Prove that the equation

$$x^{2014} - 2x^{2013} + 3x^{2012} + \dots + 2013x^2 - 2014x + 2015 = 0$$

has no real root.