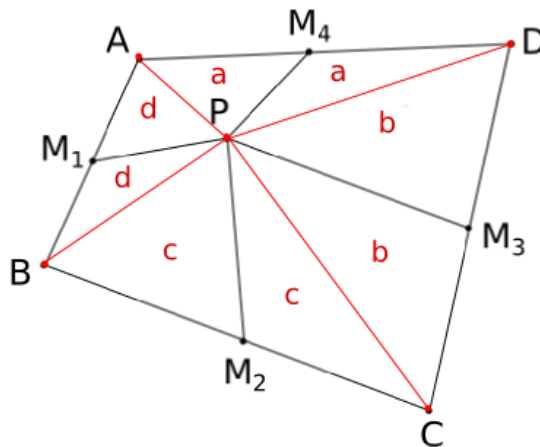


MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 10, July 31, 2017

1. There are 49 ways, and even more methods of arriving at this answer. Perhaps the easiest is to use cases starting with using 0, 1 or 2 possible 50 cent coins.
2. Divide the grid into nine 1×1 squares. If ten darts are thrown, at least one square contains at least two darts. These darts are less than $\sqrt{2}$ from each other.
3. (a) Both triangles (ABM and AMC) have the same height and base size, thus its area is the same.
(b) Note that if we divide each quadrangle in two triangles by connecting the point P with each of the original corners we can use the previous exercise.



Note that

$$a + d = 7 \text{ cm}^2, \quad a + b = 9 \text{ cm}^2, \quad b + c = 12 \text{ cm}^2 \quad \text{and} \quad d + c = X \text{ cm}^2.$$

So we have

$$a + b + c + d = (7 + 12) \text{ cm}^2 = (9 + X) \text{ cm}^2,$$

so the area we are looking for is precisely: $X = 7 + 12 - 9 = 10 \text{ cm}^2$.

4. (a) $29 = 5^2 + 2^2$, $37 = 6^2 + 1^2$. For 30, note that none of the following are square numbers:

$$301 = 29, 304 = 26, 309 = 21, 3016 = 14, 3025 = 5.$$

Similarly, 31 cannot be expressed as a sum of two squares.

(b) Easy.

(c) $1073 = (5^2 + 2^2)(6^2 + 1^2) = (302)^2 + (5 + 12)^2$.

Swapping $5^2 + 2^2$ with $2^2 + 5^2$ yields $1073 = 7^2 + 32^2$.

5. Pigeon-hole principle. Each number can be written in the form $2^k(2m + 1)$ where $k, m \geq 0$. Since each number is less than 1001, m must be less than 500.

So since you're choosing 501 numbers, two of the numbers must have the same m value.

These two numbers can be written as $2^{k_1}(2m + 1)$ and $2^{k_2}(2m + 1)$.

Either $k_1 \leq k_2$ or $k_2 \leq k_1$, so without loss of generality, assume $k_1 \leq k_2$. Then $2^{k_1}(2m + 1)$ divides $2^{k_2}(2m + 1)$, which concludes the proof.

Senior Questions

The number of ways to obtain k when rolling two dices coincides with the coefficient of x^k in

$$f(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)^2.$$

Now, note that

$$\begin{aligned} f(x) &= x^{12} + 2x^{11} + 3x^{10} + 4x^9 + 5x^8 + 6x^7 + 5x^6 + 4x^5 + 3x^4 + 2x^3 + x^2 \\ &= (x + x^2)^2(x^3 - x^2 + x)^2(x^3 + x^2 + x)^2. \end{aligned}$$

In order to find two 6-sided dice we need to be able to decompose $f(x) = B(x)C(x)$ as product of two polynomials satisfying:

$$\begin{cases} B(0) = C(0) = 0 & \text{all sides have positive numbers,} \\ B(1) = C(1) = 6 & \text{the dice has exactly 6 sides.} \end{cases}$$

The only possibility is

$$\begin{aligned} B(x) &= (x + x^2)(x^2 + x + 1)(x^2 - x + 1)^2 = x^8 + x^6 + x^5 + x^4 + x^3 + x \\ C(x) &= (x + x^2)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x. \end{aligned}$$

Thus our two dices will have sides $B = 8, 6, 5, 4, 3, 1$ and $C = 4, 3, 3, 2, 2, 1$. And this construction is unique.