MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 11, August 7, 2017

1. Let the cold tap fill the bath in c minutes and the hot tap fill the bath in h minutes.

   The given information can be summarised as follows:

$$\frac{h}{5c} + \frac{c}{5h} = \frac{5}{12} \quad \text{and} \quad \frac{5}{c} + \frac{5}{h} = \frac{7}{12}.$$

   The left equation becomes $12(h^2 + c^2) = 25ch$, and then $12(h + c)^2 = 49ch$. The right equation becomes $60(h + c) = 7ch$.

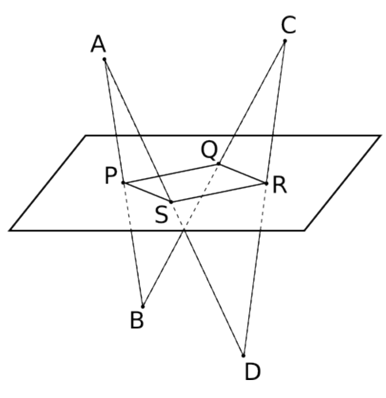
   Combining the two equations we easily obtains

$$h + c = 35 \qquad hc = 300,$$

   which gives the possible solutions $(h, c) = (20, 15)$ or $(h, c) = (20, 15)$.

   Hence one tap fills the bath in 15 minutes, the other in 20 minutes.

2. Find the midpoints P, Q, R, S of AB, BC, CD and DA respectively. The required plane is the plane through P, Q, R which we first show passes through S.

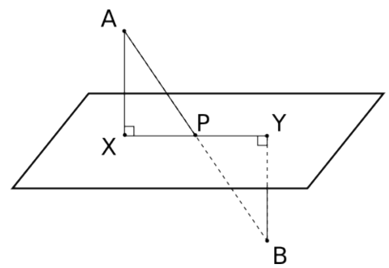


   Note that $PQ \| AC$ as P, Q are midpoints and $SR \| AC$ as S, R are midpoints so $PQ \| SR$ as well.

   Similarly $PS \| QR$ so that $PQRS$ is a parallelogram and therefore lies in a plane.

   We now will show that the perpendicular distance from A and B to the plane is the same. (In the same way it follows that the perpendicular distance from A, B, C and D is the same.)

Drop a perpendicular from A to the plane at X. The plane of AX and AP is perpendicular to the plane and so the perpendicular BY from B to the plane lies in the plane of AX and AP. We now have $|AP| = |PB|$ (since P was the midpoint by definition), $\angle APX = \angle BPY$ and by construction $\angle AXP = \angle BYP = 90°$ so triangles AXP and BPY are congruent and $|AX| = |BY|$ as required.



3. Note that we can rewrite

$$N = n^5 - 5n^3 + 4n = n(n+1)(n-1)(n+2)(n-2).$$

We need to show that $120 = 8 \cdot 3 \cdot 5$ divides N, but note that in any five consecutive integers (like $n-2, n-1, n, n+1, n+2$) we clearly can find:

- (at least) one multiple of 3,
- one multiple of 5,
- one multiple of 4 and another even number,

so after multiplying them we know that N is necessarily a multiple of 120.

4. (a) x^3 must be an integer, as $x^3 = 10 + 5[x]$ is an integer. Also $x^3 \leq 10 + 5x$ and $x^3 > 10 + 5(x-1)$ so certainly $2 < x < 3$. Hence $[x] = 2$ and therefore $x^3 = 10 + 5 \cdot 2 = 20$ and

$$x = \sqrt[3]{20} = 2.71448\ldots.$$

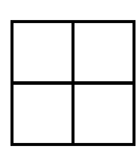
(b) $y^3 = 10 + 5\{y\}$ so $10 < y^3 < 15$ which implies that $[y] = 2$. Thus $\{y\} = y - 2$ and so

$$y^3 = 10 + 5(y-2) = 5y$$

and (since clearly $y \neq 0$) $y^2 = 5$,

$$y = \sqrt{5} = 2.23606...$$

5. Divide the board into quarters in both directions obtaining 16 "cells" each consisting of 4 mutually adjacent squares (like the one in the picture).



Of the seventeen numbers $1, 2, \ldots, 17$ there must be more than one in at least one of the cells (since we only have 16 different cells). The difference of any two of these numbers is at most 16.

6. Since the sequence is arithmetic then

$$a_n = a_1 + (n-1)d = 10 + (n-1)d,$$

where d denotes the difference of consecutive elements.

Now using the data $a_{a_2} = 100$, we have

$$100 = 10 + (a_2 - 1)d = 10 + (10 + (2-1)d)d = 10 + 10d + 2d^2,$$

or equivalently

$$d^2 + 10d - 90 = 0 \qquad d = \frac{-9 \pm \sqrt{81 + 4 \cdot 90}}{2}$$

so $d = -15, 6$ and since $d > 0$ we have that $d = 6$.

Now we can compute $a_{a_{a_3}}$:

$$\begin{aligned} a_{a_{a_3}} &= 10 + (a_{a_3} - 1)d \\ &= 10 + ((10 + (a_3 - 1)d - 1)d \\ &= 10 + ((10 + (10 + (3-1)d) - 1)d - 1)d \\ &= 10 + ((10 + (10 + 2 \times 6) - 1) \times 6 - 1) \times 6 \\ &= 10 + ((10 + (22 - 1) \times 6 - 1) \times 6 \\ &= 10 + 135 \times 6 = 820 \end{aligned}$$