MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 15, August 29, 2016

1. Since $1+2=3$ is prime, we know that $n \geq 2$. We show that $n=2$. Let $f(x)$ be the sum of n consecutive positive integers starting from x. Then

$$f(x) = x + (x+1) + (x+2) \ldots + (x+n-1) = \frac{n(2x+n-1)}{2}.$$

For every integer x, note that exactly one of n or $2x+n-1$ in the RHS of the above equation is alway even. Hence either $n/2$ is an integer, or $(2x+n-1)/2$ is an integer. Moreover, if $n>2$ then either $n/2$, or $(2x+n-1)/2$ is an integer greater than 1. Therefore, if $n>2$ then $f(x)$ is the product of two integers greater than 1 for every x, thus $f(x)$ is never prime for $n>2$.

2. Let x_1 be an odd natural number, such that the next term x_2 of the the sequence is also odd. Then the largest digit of x_1 must be even, since the sum two odd number is even. Hence, the last digit of x_1 will change by at least 2 and at most 8 and the second last digit of x_1 will change by at most 1.

Now since x_1 is odd, the last digit of x_1 is odd. Since the largest digit of x_1 is even, the last digit of x_1 can not be the largest digit of x_1. Therefore, since the second last digit of x_1 can change by at most 1, the largest digit of x_1 can change by at most 1. Therefore, if the largest digit of x_2 had been changed compare to x_1. Then, either the largest digit of x_1 changed by 1 and becomes an odd number in x_2, or the last digit of x_2 which is odd had become greater than the largest digit of x_1. Therefore, if the largest digit of x_2 had been changed compare to x_1, then the next term of the sequence x_3 becomes even.

Thus, to obtain maximum possible odd number for this sequence. The largest digit of this sequence must not change, and therefore the last digit of this sequence must change by the same amount. In addition, the last digit of the sequence must change by 2 the smallest amount possible, as to not surpass the largest digit prematurely. Hence, the maximum of terms possible is 5.

3. Yes. Consider the quadratic equation $x^2+5x+6=0$.

4. First, suppose the king moves up or right, but not diagonally. Then the king must take 5 moves right, and 5 moves up to get to the top right hand corner of the chess

board. To calculate the number m_0 of unique paths the king can take, we can think of the king picking from 10 possible moves, without caring about the order of the 5 individual up moves, or the 5 individual right moves he makes; that is

$$m_0 = \frac{10!}{5! \times 5!},$$

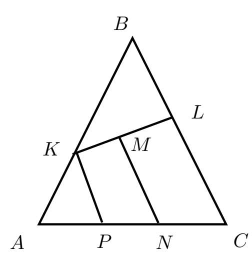
where $10! = 10 \times 9 \times 8 \ldots$ (the number of ways to order 10 objects without replacement).

Now, suppose the king takes one diagonal. Then the king must take 4 moves right, 4 moves up, and 1 mover diagonally to get to the top right hand corner of the chess board. The number m_1 of unique paths the king can take in this fashion is

$$m_1 = \frac{10!}{4! \times 4! \times 1!}.$$

Since the king can make up to 5 diagonal moves, repeating the above calculations for m_2, m_3, m_4 and m_5 then adding yields 1683 possible ways the king can move to the top right hand corner.

5. Draw a straight line KP parallel to LC, where P is a point on AC. Then $KLCP$ is a trapezoid, hence its mid-line $MN = \frac{1}{2}(KP + LC) = \frac{1}{2}(AK + LC) = \frac{1}{2}KL = KM = ML$. Therefore KL is a diameter of a circle passing through K, N, L. Thus, $\angle KNL = 90°$.

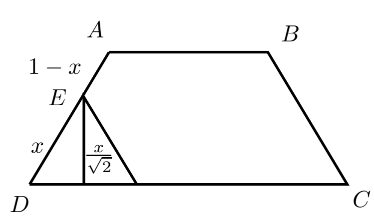


6. It is possible to show that $ABCD$ is isosceles, with base angles of $45°$. Let $DE = x$, then $AE = 1 - x$, and we obtain the equation

$$1 - x \geq x/\sqrt{2}.$$

This yields

$$x \leq 2 - \sqrt{2}.$$



Senior Questions

1. Let x_1 and x_2 be integers, such that (x_1, y_1) and (x_2, y_2) are two points on the given polynomial. Then
$$y_1 = a_0 + a_1x_1 + a_2x_1^2 + \ldots,$$
and
$$y_2 = a_0 + a_1x_2 + a_2x_2^2 + \ldots,$$
for some integers $a_0, a_1, a_2, \ldots$. Hence
$$y_1 - y_2 = a_1(x_1 - x_2) + a_2(x_1^2 - x_2^2) + \ldots. \tag{1}$$
Since $x_1^k - x_2^k$ is always divisible by $x_1 - x_2$ for all integers k, the RHS of (1) is divisible by $x_1 - x_2$. Thus, $y_1 - y_2 = n(x_1 - x_2)$ for some integer n.

   Now the distance d between (x_1, y_1) and (x_2, y_2) is
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_1 - x_2)^2(1 + n^2)}, \tag{2}$$
since $y_1 - y_2 = n(x_1 - x_2)$ for some integer n. We show that if d is an integer, then the gradient g given by
$$g = \frac{y_1 - y_2}{x_1 - x_2},$$
must be 0. Thus completing the proof.

   If d is an integer, then by (1), the expression $1 + n^2$ must be square of some integer, which implies $n = 0$. But if $n = 0$, then $y_1 - y_2 = n(x_1 - x_2) = 0$. Therefore, $g = 0$.

2. Since $(x-1)(x-2)$ is a quadratic, the remainder of the x^{2016} divided by $(x-1)(x-2)$ must be of the form $ax + b$, for some integers a, b. Hence
$$x^{2016} = (x-1)(x-2)f(x) + ax + b, \tag{3}$$
where $f(x)$ is a polynomial of degree 2014. We can find a and b by substituting $x = 1$ and $x = 2$ into (3), which gives
$$a = 2^{2016} - 1 \quad \text{and} \quad b = 2 - 2^{2016}$$

3. Call one of the people A. A corresponds with 16 others on three topics, hence at least six of these people are on the same topic, say T_1. Call these people B, C, D, E, F, G. If any two of these correspond on T_1 then A together with these two constitute a set of three who correspond with one another on the same topic. Suppose no two of B, C, D, E, F, G correspond on T_1. Then they correspond on T_2 and T_3. B corresponds with the five others on two topics, so with at least three on one topic, say with C, D, E on $_2$. If any two of C, D, E correspond on T_2, then B together with these two constitute a set of three who correspond with one another on the same topic. Otherwise, C, D, E all correspond on T_3 and constitute a set of three who correspond with one another on the same topic.