MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 14, September 4, 2017

1. The sum of all numbers in the circle is $1+2+3+4+5+6+7+8+9=45$.

   Say the numbers, in clockwise order, are $0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9$. Then

   $$45 = (a_1+a_2+a_3)+(a_4+a_5+a_6)+(a_7+a_8+a_9),$$

   so at least one of the three triplets has sum at least 15. A possible arrangement such that no triplet has sum larger than 15 is $0, 6, 2, 7, 3, 4, 8, 1, 5, 9$.

2. consider $x = 1-t, y = 1+t$ with $t \in [0,1]$. Then

   $$\begin{aligned}
   x^2y^2(x^2+y^2) &= (1-t)^3(1+t)^3\left((1-t)^3+(1+t)^3\right)\\
   &= (1-t^2)^3(2+6t^2)\\
   &= 2(1-t^2)^3(1+3t^2)\\
   &= 2(1-3t^2+3t^4-t^6)(1+3t^2)\\
   &= 2(1+3t^2-3t^2-9t^4+3t^4+9t^6-t^6+t^8)\\
   &= 2(1-6t^4+8t^6-3t^8)\\
   &= 2(1-[2(t^2-1)^2+2t^4+t^6])
   \end{aligned}$$

   So clearly,
   $$x^2y^2(x^2+y^2) \le 2$$
   if and only if
   $$2(t^2-1)^2+2t^4+t^6 > 0, \qquad \text{for } 0 < t < 1.$$
   But this is true since it is a sum of squares, therefore positive.

3. Clearing denominators and manipulating the expression, we get

   $$\begin{aligned}
   (r+q)^2(n+1) &= (r+qn)(q+rn)\\
   &= rq + r^2n + q^2n + rqn^2\\
   &= rq(n^2+1) + (r^2+q^2)n\\
   &= rq[(n+1)(n-1)+2] + (r^2+q^2)[(n+1)-1]\\
   &= rq(n+1)(n-1) + (r^2+q^2)(n+1) - (q-r)^2
   \end{aligned}$$

that can be expressed as

$$(r+q)^2 = r^2 + q^2 + qr(n-1) - \frac{(q-r)^2}{n+1}. \tag{1}$$

Note that from (1) we have

$$\begin{aligned} r^2 + 2qr + q^2 &= r^2 + q^2 + qrn - qr - \frac{(q-r)^2}{n+1} \\ qr(3-n) &= -\frac{(q-r)^2}{n+1}, \end{aligned}$$

in other words:

$$(n+1)(n-3) = \frac{(q-r)^2}{qr}. \tag{2}$$

Also, manipulating (1) we have that

$$\begin{aligned} qr(n-1) &= 2qr + \frac{(q-r)^2}{(n+1)} \\ (n-1)(n+1) &= 2(n+1) + \frac{(q-r)^2}{qr} \\ (n-1)^2 &= \frac{(q+r)^2}{qr} \end{aligned} \tag{3}$$

Combining identities (2) and (3) we have that

$$\frac{n-3}{n+1} = \frac{(n-3)(n+1)}{(n-1)^2}\left(\frac{(n-1)}{(n+1)}\right)^2 = \left(\frac{(q-r)(n-1)}{(q+r)(n+1)}\right)^2$$

Thus, $\sqrt{\frac{n-3}{n+1}} = \left|\frac{(q-r)(n-1)}{(q+r)(n+1)}\right|$ is a rational number.

4. Note that

$$|x-1| = \begin{cases} x-1 & \text{if } x \geq 1, \\ 1-x & \text{if } x < 1, \end{cases}$$

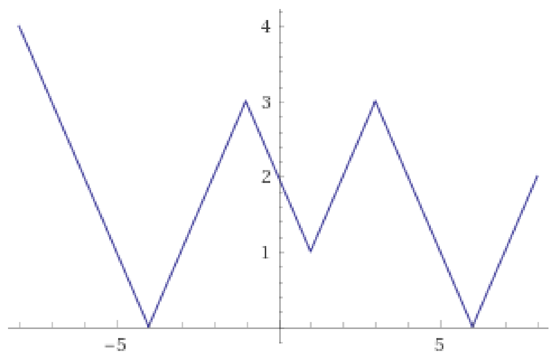
Which implies that,

$$y = \begin{cases} ||x-3|-3| & \text{if } x \geq 1, \\ ||x+1|-3| & \text{if } x < 1. \end{cases}$$

Working separately with the expressions $||x-3|-3|$ and $||x+1|-3|$ in the same fashion we arrive to 6 different cases:

$$y = \begin{cases} x-6 & \text{if } 6 \leq x \geq 8, \\ 6-x & \text{if } 3 \leq x < 6, \\ x & \text{if } 1 \leq x < 3, \\ 2-x & \text{if } -1 \leq x < 1, \\ x+2 & \text{if } -2 \leq x < -1, \\ -x-2 & \text{if } -8 \leq x < -2, \end{cases}$$

With graphic representation:



5. Let's work in $[0, 2\pi]$.

   (a) Square both sides to get $\sin(2x) > 0$. So $0 < x < \pi/2$.

   (b) Square both sides to get $2|\sin x|\cos x > 0$. So $0 < x < \pi/2$ or $3\pi/2 < x < 2\pi$.

6. Setting $y = 0$ in the equation, we get $f(x^3) = xf(x^2)$ which implies that

$$f(x^3 + y^3) = f(x^3) + f(y^3)$$

The problem now is equivalent to finding all real functions with

$$f(x + y) = f(x) + f(y) \quad \text{and} \quad f(x^3) = xf(x^2).$$

Let then $f(1) = a \in \mathbb{R}$ and note that

$$\begin{aligned} f\left((x+1)^3\right) &= f(x^3 + 3x^2 + 3x + 1) \\ &= f(x^3) + 3f(x^2) + 3f(x) + a \\ &= xf(x^2) + 3f(x^2) + 3f(x) + a = (x+3)f(x^2) + 3f(x) + a. \end{aligned}$$

But also $f\left((x+1)^3\right) = (x+1)f((x+1)^2) = (x+1)(f(x^2) + 2f(x) + a)$. Comparing these two identities we obtain

$$f(x^2) = xf(x) + \frac{ax - f(x)}{2}.$$

Then, using this last equality, we have

$$f(x^6) = f\left((x^3)^2\right) = x^3 f(x^3) + \frac{ax^3 - f(x^3)}{2} = x^4 f(x^2) + \frac{ax^3 - xf(x^2)}{2}$$

But also, using $f(x^3) = xf(x^2)$, we have

$$f(x^6) = f\left((x^2)^3\right) = x^4 f(x^2) + \frac{ax^4 - x^2 f(x^2)}{2},$$

and hence $ax^3 - xf(x^2) = ax^4 - x^2 f(x^2)$. This implies that $f(x^2) = ax^2$ for every $x \notin \{0, 1\}$ and so $f(x) = ax$ for every $x \neq 1, 0$.

This is still true for $x = 1$ (definition of a). And since $f(x + y) = f(x) + f(y)$ implies $f(0) = 0$ and $f(-x) = -f(x)$, we get

$$f(x) = ax \text{ for every } x \in \mathbb{R}$$

which, indeed, is the solution we were looking for.

Senior Questions

1. We have $f(n) = n^2 + 1$ and $g(n) = n(n+1) = n^2 + n$, which means that

$$F(n) = \sum_{k=1}^{n} k^2 + 1 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 1 = \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n+1)(2n+1)+6n}{6}$$

Similarly,

$$G(n) = \sum_{k=1}^{n} k^2 + k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k = \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} = \frac{n(n+1)(2n+4)}{6}$$

The limit is then

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)+6n}{n(n+1)(2n+4)} = 1$$

by L'Hôpital's Rule.

2. Let r_1 and R_1 the radius of inscribed, respective circumscribed, circle of $\triangle ABC$. In that case $r_1 = \frac{l\sqrt{3}}{6}$ and $R_1 = \frac{l\sqrt{3}}{3}$.

The distance between the plane of the triangle and the centre of the sphere is $d = \sqrt{r^2 - r_1^2}$. The distance between any vertex of the triangle and the centre of the sphere is $D = \sqrt{d^2 + R_1^2} = \sqrt{r^2 - r_1^2 + R_1^2} = \sqrt{r^2 + \frac{l^2}{4}}$.

3. We will need the auxiliary result.

Lemma: *Let $f(x)$ be a continuous on $[a, b]$ and differentiable on (a, b) function. Suppose that $f(x)$ has in $[a, b]$ k distinct real roots. Then, for arbitrary real c, function $g(x) = cf(x) + f'(x)$ has on $[a, b]$ at least $k - 1$ distinct real roots.*

Proof. Consider function $h(x) = e^{cx} f(x)$. It easy to see that $h(x)$ satisfy all conditions of Rolle theorem on $[a, b]$. Hence $h'(x)$ has on $[a, b]$ at least $k - 1$ distinct real roots. Note that $e^{cx} \neq 0$ and $h'(x) = e^{cx}(cf(x) + f'(x))$, as we wanted to proof. □

In our case we have $m^2 > 4n$ so equation $t^2 - mt + n = 0$ has two distinct roots: t_1 and t_2. Let $a = x_1$ and $b = x_3$.

The function $f(x)$ has on $[a, b]$ three real roots, so it follows from the Lemma that the function

$$f_1(x) = t_1 f(x) + f'(x)$$

has at least two distinct roots on $[a, b]$. Applying once more the Lemma for $f_1(x)$ and $c = t_2$ we obtain that

$$f_2(x) = (t_1 t_2 f(x) + t_2 f'(x)) + t_1 f'(x) + f''(x)$$

has at least one real root on $[a, b]$.

Now recall that t_1, t_2 were roots of $t^2 - mt + n = 0$, thus

$$t_1 + t_2 = m \text{ and } t_1 t_2 = n.$$

In other words, $f''(x) + mf'(x) + nf(x) = 0$ has on $[x_1, x_3]$ at least one real root.