MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 10, July 30, 2013

1.

$$\begin{aligned}(x^{-1}+y^{-1})^{-1} &= \frac{1}{\frac{1}{x}+\frac{1}{y}} \\ &= \frac{1}{\frac{y+x}{xy}} \\ &= \frac{xy}{y+x}.\end{aligned}$$

2. For a number to be a cube it's prime factorisation must contain only cubes. The prime factorisation of $60 = 2^2 \times 3 \times 5$ so $n = 2 \times 3^2 \times 5^2 = 450$.

3. Using long division we can see that 12 950 264 876 is divisible by 3 but that $4\,650\,088\,292 = \frac{12\,950\,264\,876}{3}$ is not. Thus the prime factorisation of 12 950 264 876 contains a 3 which is not squared, so cannot be a square.

4. There is a multiplier α such that the angles of our triangle are 2α, 3α and 4α. Using the angle sum $2\alpha + 3\alpha + 4\alpha = 180 \implies \alpha = 20$. So the angles are 40, 60 and 80.

5. Let the median from C to AB meet AB at D. Since DC has length $\frac{1}{2}AB$, both triangles ADC and BDC are isoceles with $AD = DC$ and $DC = DB$. So $\angle DCA = \angle DAC = \alpha$, $\angle DCB = \angle DBC = \beta$ and $\angle DAC + \angle DBC + \angle ACB = \alpha + \beta + (\alpha + \beta) = 180 \implies \alpha + \beta = 90$. Since $\angle ACB = \alpha + \beta = 90$.

6. (a) Recall that $\gcd(a + mb, b) = \gcd(a, b)$. So if we have $\gcd(m, n)$ with $m > n$ and we divide m by n to get a remainder r, then $\gcd(m, n) = \gcd(n, r)$. So divide $2^{50} + 1$ by $2^{20} + 1$ and we get

$$2^{50} + 1 = (2^{20} + 1)(2^{30} - 2^{10}) + 2^{10} + 1$$

so

$$\gcd(2^{50} + 1, 2^{20} + 1) = \gcd(2^{20} + 1, 2^{10} + 1).$$

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Now we divide $2^{20}+1$ by $2^{10}+1$, and continue dividing the larger by the smaller and replacing the larger with the remainder, so it goes

$$\begin{aligned}\gcd(2^{20}+1, 2^{10}+1) &= \gcd(2^{10}+1, 2)\\ &= \gcd(2,1)\\ &= \gcd(1,0)\\ &= 1.\end{aligned}$$

(b) Note that for odd powers n, the remainder when dividing 2^n by 3 is 2, whereas for even powers n the remainder is 1. The remainder when dividing 1 by 3 is 1. Since the sum of the remainders of $2^n/3$ for odd n, and $1/3$ is 3, then 3 divides 2^n+1 for odd n. So the greatest common divisor of 2^n+1 and 2^m+1 for odd n and m must be at least 3.

Senior Questions

1. (a) The surface area of a surface of revolution constructed by rotating the graph $y = f(x)$ about the x-axis for $a \le x \le b$ is given by

$$A = 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\, dx.$$

Since Gabriel's horn is infinitely long we actually mean

$$A = \lim_{n\to\infty} 2\pi \int_1^n \frac{1}{x}\sqrt{1+\left(-\frac{1}{x}\right)^2}\, dx.$$

Note that $\frac{1}{x}\sqrt{1+\frac{1}{x^4}} \ge \frac{1}{x} > 0$ for $x \ge 1$ so

$$\int_1^n \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx > \int_1^n \frac{1}{x}\, dx = \ln n.$$

Since $\ln n \to \infty$ as $n \to \infty$ so does $\int_1^n \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\, dx$ and A is infinitely large.

(b) The volume of a solid of revolution constructed by rotating $y = f(x)$, $a \le x \le b$ about the x-axis is given by

$$V = \pi \int_a^b f(x)^2\, dx.$$

For Gabriel's horn then

$$\begin{aligned}V &= \pi \lim_{n\to\infty} \int_1^n \frac{1}{x^2}\, dx\\ &= \pi \lim_{n\to\infty} \left[-\frac{1}{x}\right]_1^n\\ &= \pi \lim_{n\to\infty} \left(1-\frac{1}{n}\right)\\ &= \pi.\end{aligned}$$

So if Gabriel wanted to paint the infinite surface area of the inside of his infinitely long horn he'd need at most π units of paint. Wait, what?

2. Using various angle expansions obtain

$$\begin{aligned}
\cos((n+2)\theta) &= \cos((n+1)\theta)\cos\theta - \sin((n+1)\theta)\sin\theta \\
&= \cos((n+1)\theta)\cos\theta - \sin\theta(\sin(n\theta)\cos\theta + \cos(n\theta)\sin\theta) \\
&= \cos((n+1)\theta)\cos\theta - \sin(n\theta)\sin\theta\cos\theta - \cos(n\theta)\sin^2(\theta) \\
&= \cos((n+1)\theta)\cos\theta - \frac{1}{2}\sin(n\theta)\sin(2\theta) - \cos(n\theta)\sin^2\theta.
\end{aligned}$$

Now note

$$\begin{aligned}
\sin(n\theta)\sin(2\theta) &= \cos(n\theta)\cos(2\theta) - \cos(n\theta + 2\theta) \\
&= \cos(n\theta)(2\cos^2\theta - 1) - \cos((n+2)\theta).
\end{aligned}$$

So

$$\begin{aligned}
\cos((n+2)\theta) &= \cos((n+1)\theta)\cos\theta - \frac{1}{2}(\cos(n\theta)(2\cos^2\theta - 1) - \cos((n+2)\theta)) - \cos(n\theta)\sin^2\theta \\
&= \cos((n+1)\theta)\cos\theta - \cos(n\theta)\cos^2\theta + \frac{1}{2}\cos(n\theta) + \frac{1}{2}\cos((n+2)\theta) - \cos(n\theta)\sin^2\theta \\
\frac{1}{2}\cos((n+2)\theta) &= \cos((n+1)\theta)\cos\theta + \frac{1}{2}\cos(n\theta) - \cos(n\theta)(\cos^2\theta + \sin^2\theta) \\
&= \cos((n+1)\theta)\cos\theta - \frac{1}{2}\cos(n\theta) \\
\cos((n+2)\theta) &= 2\cos((n+1)\theta)\cos\theta - \cos(n\theta).
\end{aligned}$$

Iteratively expanding $\cos 5\theta$ we obtain

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.$$