MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 12, August 13, 2013

1. Start with

$$\begin{aligned}\frac{x+3y}{2x+5y} &= \frac{4}{7}\\ 7(x+3y) &= 4(2x+5y)\\ y &= x\end{aligned}$$

   but we mustn't have $2x+5y=0$, which means $x \neq -\frac{5}{2}y$. The final solution then is $x = y \neq 0$.

2. Solving simultaneously we get $a=0$, $b=6$, $c=7$, $d=3$ and $e=-1$, so c is the largest. Can you do this without solving simultaneously?

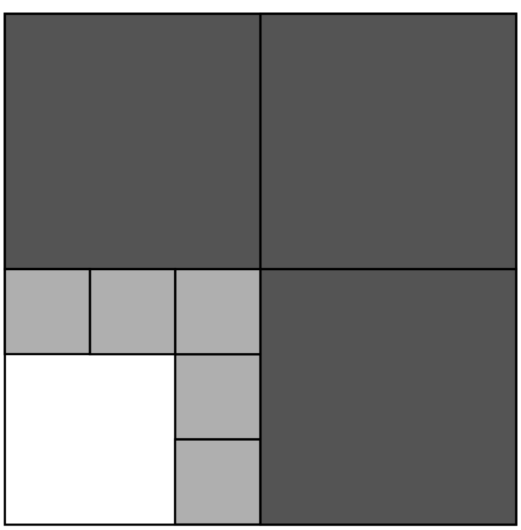

Figure 1: Solution for Question 3

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Solution to question 4 provided by G. Liang

3.

4. Begin by constructing the equilateral triangle ADB. Draw the line CD to intersect AB at E. Draw EG parallel to DB and EF parallel to DA. Connect F and G then triangle EFG is equilateral. To prove this is true, construct $B'A'$ parallel to BA and passing through D, and show that $BB'D$ and $AA'D$ are similar to GBE and FAE respectively.

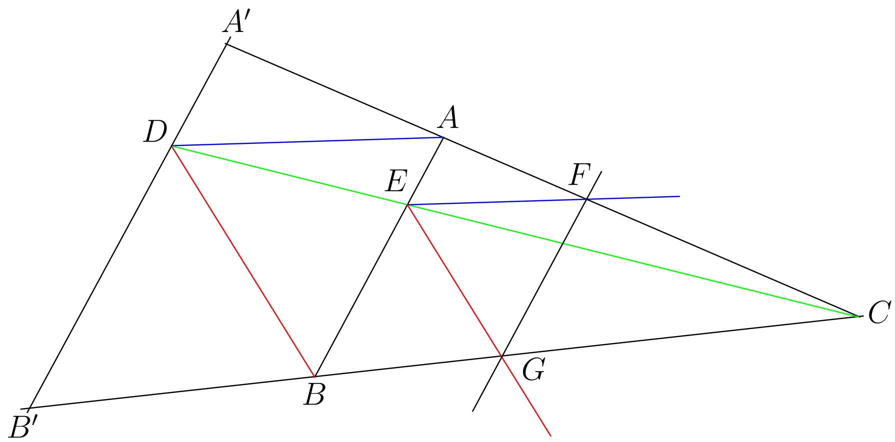

Figure 2: Solution for Question 4

5. Suppose otherwise, i.e., there is a 50×50 corner with no queens on it. Let's suppose the empty corner is the bottom-left (the argument will hold if any corner is chosen). To fit 100 queens on the chess board, all not attacking each other, we must at least have 1 queen per row, and 1 queen per column. Since the bottom-left corner is empty, the 50 top rows must have a queen each, and the 50 left columns must also have a queen each. Since none are attacking each other, there can be no queens in the top-right (otherwise not all rows/columns would have exactly one queen). Now we count the number of diagonals that stretch from the top-left 50×50 corner to the bottom-right 50×50 corner; there are 99, and hence not enough diagonals for one per queen, and so two queens must share a diagonal. This is a contradiction, so we cannot have an empty corner.

6. (a) Just expand the two right-hand sides:

$$\begin{aligned}(x^3-1)(x^{12}+x^9+x^6+x^3+1) &= x^{15}-x^{12}+x^{12}-x^9+x^9-x^6+x^6-x^3+x^3-1\\ &= x^{15}-1\\ (x^5-1)(x^{10}+x^5+1) &= x^{15}-x^{10}+x^{10}-x^5+x^5-1\\ &= x^{15}-1.\end{aligned}$$

(b) From part (a) $2^{15}-1$ has factors $2^5-1=31$ and $2^3-1=7$, both of which are prime. We can use long division to divide $2^{10}+x^5+1$ by 2^3-1, which gives us the remainder of dividing $2^{15} - 1$ by both $2^5 - 1$ and $2^3 - 1$. So the prime factorisation is $2^{15} - 1 = 31 \times 7 \times 151$.

(c) We can use

$$\begin{aligned} x^{15} + 1 &= (x^3 + 1)(x^{12} - x^9 + x^6 - x^3 + 1) \\ &= (x^5 + 1)(x^{10} - x^5 + 1). \end{aligned}$$

So we can write

$$\begin{aligned} 2^{15} + 1 &= (2^5 + 1)(2^3 + 1)R \\ &= 33 \times 9 \times R. \end{aligned}$$

Using long division we get $R = 138 + \frac{1}{3}$ so

$$\begin{aligned} 2^{15} + 1 &= 3^3 \times 11 \times \left(\frac{3 \times 138 + 1}{3} \right) \\ &= 3^2 \times 11 \times (3 \times 110 + 1) \\ &= 3^2 \times 11 \times 331. \end{aligned}$$