MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 14, August 27, 2013

1. (a) Assuming $n > 0$ is positive

$$n^2 + n < n^2 + 2n + 1 = (n+1)^2$$

and

$$n^2 < n^2 + n.$$

So $n^2 < n^2 + n < (n+1)^2$ so $n^2 + n$ can't be a square. If $n < -1$

$$n^2 + 2n + 1 < n^2 + n < n^2,$$

so similarly $n^2 + n$ is not a square

(b) Note that, if $n > 1$

$$\left(n^2 + \frac{n}{2}\right)^2 = n^4 + n^3 + \frac{n^2}{4} < n^4 + n^3 + n^2 + n$$

and

$$\left(n^2 + \frac{n}{2} - 1\right)^2 = n^4 + n^3 + \frac{9}{4}n^2 + n + 1 > n^4 + n^3 + n^2 + n.$$

2. The sum of the digits $1, 2, \ldots, 9$ is 45 $((1+9) + (2+8) + \cdots + 5)$. Also recalling that if we have a sum like $\sum_{k=0}^{n} a + k = a(n+1) + \sum_{k=0}^{n} k$, then the required sum is

$$\begin{aligned}\sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}\sum_{d=0}^{9}(a+b+c+d) &= \sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9}\left(10\,(a+b+c) + \sum_{d=0}^{9} d\right) \\ &= \sum_{a=0}^{9}\sum_{b=0}^{9}\sum_{c=0}^{9} 45 + 10a + 10b + 10c\end{aligned}$$

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Some problems from the Tournament of Towns

$$
\begin{aligned}
&= \sum_{a=0}^{9}\sum_{b=0}^{9} 10(45 + 10a + 10b) + 10\sum_{c=0}^{9} c \\
&= \sum_{a=0}^{9}\sum_{b=0}^{9} 450 + 100a + 100b + 10 \times 45 \\
&= \sum_{a=0}^{9} 10(900 + 100a) + 100\sum_{b=0}^{9} b \\
&= \sum_{a=0}^{9} 9\,000 + 1\,000a + 100 \times 45 \\
&= 10 \times 13\,500 + 1\,000 \times 45 \\
&= 180\,000.
\end{aligned}
$$

3. First consider the case where an equilateral triangle has its base aligned with the grid. If the base takes $2n$ grid spaces, the triangle is $n\sqrt{3}$ grid spaces high, so its apex won't be on a vertex.

   Now consider instead that the triangle's base makes an angle of θ with the grid, and has sides lengths x. For each of the triangle's points to lie on a vertex we require that both $\tan\theta = r$ is rational and $\tan\left(\theta + \frac{\pi}{3}\right)$ is rational. Now

$$
\begin{aligned}
\tan\left(\theta + \frac{\pi}{3}\right) &= \frac{\tan\theta + \tan\frac{\pi}{3}}{1 - \tan\theta\tan\frac{\pi}{3}} \\
&= \frac{r + \sqrt{3}}{1 - r\sqrt{3}} \\
&= \frac{(r + \sqrt{3})(1 + r\sqrt{3})}{1 - 3r^2} \\
&= \frac{r + 3r + (1 + r^2)\sqrt{3}}{1 - 3r^2},
\end{aligned}
$$

   which is not rational because $\sqrt{3}$ isn't.

4. If we take the prime factorisation of the numbers 2 through 10, we get

$$2, 3, 2^2, 5, 2 \times 3, 7, 2^3, 3^2, 2 \times 5.$$

   So the smallest number divisble by all these numbers is the number with prime factorisation made by taking as few as these as possible so that each factorisation above is included: $2^3 \times 3^2 \times 5 \times 7 = 2520$ gives our answer.

5. We can do this in 25 extra boxes (50 in total). Number the checkers 1 through 25, then these are the two moves to use:

   - If second last checker is smaller than the last checker, add a box so it can jump over. Then have the 4th last jump the 3rd last, the 6th last jump the 5th last and so on.

- If the second last checker is larger than the last checker, move the last checker over so that the 3rd last can jump the second last. Then have the 5th last jump the 4th last, the 7th last jump the 6th last and so on.

Here is an example with 5 checkers:

1	2	3	4	5					
1		3	2	5	4				
		3	1	5	2	4			
		3		5	1	4	2		
				5	3	4	1	2	
				5		4	3	2	1
					5	4	3	2	1

6. The trick here is to rotate the triangle about A by $60°$.

Since $\angle OAC = \angle O'A'C'$ we see that $\angle OAO' = 60°$, so triangle OAO' is equilateral, meaning triangle $OO'B$ is the triangle constructed from the 3 sides of length $|AO|$, $|BO|$ and $|CO|$. Then $\angle BOO' = 360° - 60° - x - y = 300° - x - y$, $\angle OO'B = y - 60°$ and $\angle O'BO = 2y + x - 180°$.

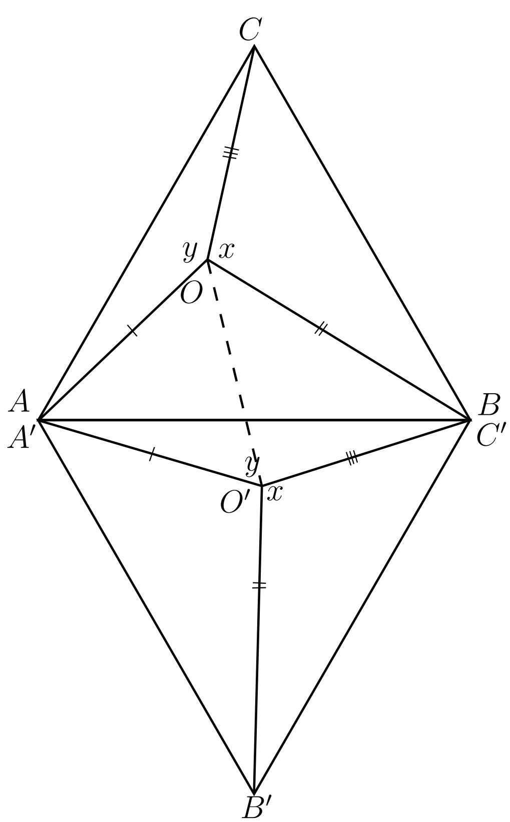

Figure 1: Picture accompanying question 6.