MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 17, September 17, 2013

1. There are 5 odd integer digits to choose from. Once one is chosen for the first digit, only 4 remain, then 3. So there are $5 \times 4 \times 3 = 60$ 3 digit numbers with distinct odd digits.

2. Squaring palindromic numbers is a good way to find new palindromic squares, provided that the digits are small enough that there's no need to do any "carrying" when we do the multiplication. So $121^2 = 14641$, $22^2 = 484$. It turns out there are no 4-digit palindromic squares.

3. Let the isoceles triangle be ABC with base BC. The square is bisected by the alitude of the triangle through A, which meets BC at D. Let E be the vertex of the square on BC between B and D and let F be the vertex of the square above E. Then triangles ABD and FBE are similar, so, letting the side length of the square be x, we get the relation
$$\frac{x}{\sqrt{10^2 - 6^2}} = \frac{6 - \frac{x}{2}}{6}$$
the solution of which is
$$x = 4.8.$$

4. We wish to find n, such that for some q_1, q_2, q_3 and r we have
$$364 = nq_1 + r, \qquad 414 = nq_2 + r \text{ and}$$
$$539 = nq_3 + r.$$
Combining the first two means
$$(q_2 - q_1)n = 414 - 364 = 50.$$
Since n and all the q's are integers, n must be a factor of 50, which are 50, 25, 10, 5, 2 or 1. Dividing 364 or 414 by 50 gives a remainder of 14, whilst dividing 539 by 50 gives a remainder of 39, so n is not 50. Dividing 364 or 414 by 25 still gives a remainder of 14, and so does dividing 539 by 25. So $n = 25$ works, and since it is larger than the other factors of 50 it is our answer.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

5. (a) $a_6 = 6 + 5 + 4 + 3 + 2 = 20$
   (b) a_n is simply the sum of integers from 2 to n, which is an arithmetic series, so $a_n = \frac{n-1}{2}(n + 2)$.
   (c) $b_6 = 6^2 + 5^2 + 4^2 + 3^2 + 2^2 = 90$
   (d) Some may recognize that b_n is the nth square pyramidal number (`en.wikipedia.org/wiki/Square_pyramidal_number`) minus 1. The formula for the nth square pyramidal number is $\frac{n}{6}(n + 1)(2n + 1)$, so $b_n = \frac{n}{6}(n + 1)(2n + 1) - 1$.

6. (a) $ABCB_1$ is a parallelogram since BC is parallel to AB_1 and CB_1 is parallel to AB. Similarly CBC_1A is a parallelogram. So now we know that A is the midpoint of B_1C_1.

   Now $\angle B_1AC = \angle ACB$ because they are alternate. If D is the point at which the altitude from A meets BC then $\angle DAC = 90 - \angle ACD = 90 - \angle ACB$ so $\angle DAC + \angle\angle B_1AC = 90$, and AD is the perpendicular bisector of B_1C_1.
   (b) Since all the altitudes are also perpendicular bisectors of a triangle, and perpendicular bisectors of a triangle are concurrent, these altitudes are also.

7. P must be on the opposite side of the chord AB from O otherwise the angle with be zero. Instead, let the angle at P be θ, then the angle at O is $180 - 2\theta$. Setting these equal gives $\theta = 180/3 = 60°$.