MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 2, May 13, 2014[1]

1. (a) To begin, $n^5 - 5n^3 + 4n$ can be factored to $(n+2)(n+1)n(n-1)(n-2)$. You can make the argument here that one of each of these factors must be divisible by one of each of 5,4,3 and 2, and so must be divisible by 120. I thought it was nice to re-write these factors as

$$\frac{(n+2)!}{(n-3)!} = 5!\binom{n+2}{5}$$

and seeing as $5! = 120$, it must be divisible by 120. Note also that for $n = 0, 1, 2$, $n^5 - 5n^3 + 4n = 0$ which is divisible by every number.

(b) Let's write $n^2 + n + 2 = (n+4)^2 - 7(n+2)$ and suppose that it is divisible by 49. If this is the case, then both $(n+4)^2$ and $7(n+2)$ must be divisible by 49, or both $(n+4)$ and $(n+2)$ by 7. This is not possible.

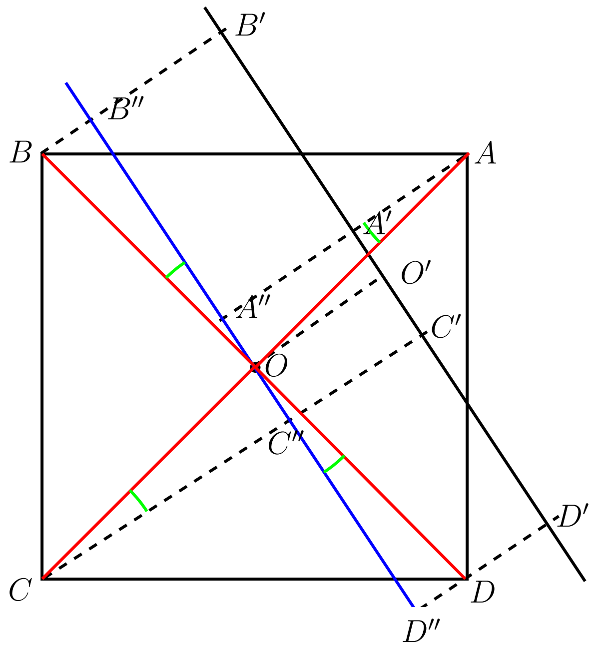
2. If A was truthful about B coming second, then B must be lying about A coming second and C about B coming third, so the order would be ABC.

If A was truthful about C coming first, then B must be lying that C was third and C about A coming first, so the order would be CAB. Either way A beat B.

3. If a number is written in its prime factorisation $n = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}$ then for it to be powerful each of the $m_i \geq 2$ and for it to be a perfect power all $m_i = c$, a constant. Thus for n to be powerful but not a perfect power all the m_i must be greater than 2, but not all the same. The smallest then, would be $2^3 \times 3^2 = 72$.

4. Draw in the lines shown in red (the diagonals of the square) and the line shown in blue (parallel to $B'D'$ through O), and label the respective points on the blue line A'', B'', C'' and D''. It can be shown (but I'll leave it out here) that the angles drawn in green are equal, let's also call them α. Since they are diagonals, the lengths of BO, AO, DO and CO are $\sqrt{2}$. So $AA'' = OA\cos\alpha = \sqrt{2}\cos\alpha$, $CC'' = \sqrt{2}\cos\alpha$, $BB'' = \sqrt{2}\sin\alpha$ and $DD'' = \sqrt{2}\sin\alpha$. If $OO' = x$ then $AA' = \sqrt{2}\cos\alpha - x$, $CC' = \sqrt{2}\cos\alpha + x$, $BB' = \sqrt{2}\sin\alpha + x$ and $DD' = x - \sqrt{2}\sin\alpha$. Substituting into $AA' \times CC' = BB' \times DD'$ we determine that $2\cos^2\alpha - x^2 = x^2 - 2\sin^2\alpha$ or $2x^2 = 2$ so $x = 1$.

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*



5. Start by factorising $x^3 - y^3 = (x-y)(x^2+xy+y^2)$ and $1729 = 7 \times 13 \times 19$. The second factor, x^2+xy+y^2, can't be factorised further, so this gives us three possiblities:

$$\begin{gathered} x-y=7 \text{ and } x^2+xy+y^2=247 \\ x-y=13 \text{ and } x^2+xy+y^2=133 \text{ or} \\ x-y=19 \text{ and } x^2+xy+y^2=91. \end{gathered}$$

For each of these, square the first and subtract it from the second to get two equations for x and y:

$$\begin{aligned} x-y&=7 & xy&=66 \\ x-y&=13 & xy&=-12 \\ x-y&=19 & xy&=-90. \end{aligned}$$

There are no possible solutions to the first, the second permits $(-1,12)$ and $(1,-12)$ and the last permits $(-9,10)$ and $(9,-10)$.

6. Start with any 9 digit number. We can make this a 10 digit number by adding a digit to the right of it. Suppose for each 9 digit number we apend the unique digit so that the sum of its digits is a multiple of 10. Suppose two numbers obtained this way are neighbours, then the sum of their digits are $d_1+d_2+\cdots+d_{10}=10n$ and $d_1+\cdots+d_i'+\cdots+d_{10}=10n+(d_i'-d_i)$, where $d_i \neq d_i'$. This second sum cannot be a multiple of 10, so no two numbers obtained in this way are neighbours. We can generate 9×10^8 numbers in this way, so there are at least 9×10^8 numbers in the collection of ten digit numbers of which none are neighbours.

   However, if we begin with all the 10 digit numbers (9×10^9 of them), and note that for each 9 digit number, we can attach one extra digit to make 10 numbers which are all neighbours. So at most there are $9 \times 10^9/10$ numbers in the collection of ten digit numbers of which none are neighbours.

So we know that at a maximum there are 9×10^8 numbers in our collection, and we have an example of a sub-collection that has this many, so the largest possible collection contains 9×10^8 numbers!