MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 10, 2014[1]

1. Let x be a natural number, then we wish to know whether

$$x^3 + (x+1)^3 + (x+2)^3$$

is divisible by 18. Letting the above sum be n

$$\begin{aligned} x^3 + (x+1)^3 + (x+2)^3 &= n \\ x^3 + (x^3 + 3x^2 + 3x + 1) + (x^3 + 6x^2 + 12x + 8) &= n \\ 3x^3 + 9x^2 + 15x + 9 &= n \\ 3(x^3 + 3x^2 + 5x + 3) &= n. \end{aligned}$$

For n to be divisible by 18, $n = 0 \mod 18$, so $x^3 + 3x^2 + 5x + 3 = 0 \mod 6$. Here we can simply try the 6 values for x and see which equal 0. Let $p(x) = x^3 + 3x^2 + 5x + 3 \mod 6$, then

$$\begin{aligned} p(0) &= 3 \mod 6 = 3, \\ p(1) &= 12 \mod 6 = 0, \\ p(2) &= 8 + 12 + 10 + 3 \mod 6 = 2 + 0 + 4 + 3 \mod 6 \\ &= 9 \mod 6 = 3 \mod 6, \\ p(3) &= 27 + 27 + 15 + 3 \mod 6 = 3 + 3 + 3 + 3 \mod 6 = 0 \mod 6, \\ p(4) &= 64 + 48 + 20 + 3 \mod 6 = 4 + 0 + 2 + 3 \mod 6 = 3 \mod 6, \\ p(5) &= 125 + 75 + 25 + 3 \mod 6 = 5 + 3 + 1 + 3 \mod 6 = 0 \mod 6. \end{aligned}$$

So n is divisible by 18 whenever $x = 1, 3, 5 \mod 6$, or in other words, whenever x is odd.

2. This problem seems counter intuitive, since on face level it looks like the contestant has two options, one of which yields the prize. But in fact, the contestant has more information to work with, due to the fact that Monty will always open a door with a goat.

Suppose the contestant first selects the door with the car, which happens with probability 1/3. Here, swapping guarantees you lose, and staying guarantees you win. If,

[1] Some problems from UNSW's publication *Parabola*

on the other hand, the contestant first picks a door with a goat, which happens with probability 2/3, then swapping guarantees the car, and staying guarantees a goat. All in all, swapping gives you the win 2 out 3 times while staying only 1 out of 3, so the better strategy is to swap when asked.

3. Clearly we cannot have 4 rows of 4 counters, as this means we have 16 on the grid, not 10. We cannot have 3 rows of 4 as this would mean the columns have 3 in them, additionally we'd have 12 counters, not 10. We cannot have 2 rows of 4, even though this satisfies the condition that rows and columns must have exactly 2 or 4 counters, since this would mean we have to few counters on the grid, and adding any will violate the above condition by producing a column of 3. Finally, we cannot have zero rows of 4 as then we would have only 8 counters, not 10. So, we must have precisely one row of 4.

   We can repeat the above, but exchanging "rows" for "columns" and we see we must also have exactly one column of 4. Since the row of 4 and column of 4 will cross, this takes up 7 counters, so we have 3 to spare. We also have 3 rows each and 3 columns each left which only have 1 counter in them, so add these last 3 so that they each occupy a different row-column pair and we have our solution.

   As for the number of solutions, choose 1 from 4 possible choices to place your row of 4, 1 from 4 possible choices for your column of 4. For the last 3 counters, it is equivalent to the number of ways of ordering the 3 counters by row (each counter is different, as they can be labelled by their column), so there are 3! ways. The total number of solutions then is $4 \times 4 \times 3!$.

   If we want to count the solutions up to a rotation, then the above counts each solution 4 times, since there are 4 ways of rotating the square. So in this case the number of solutions is $4 \times 3!$.

4. We can draw this party as a graph (not the x-axis/y-axis kind, but a collection of points, called nodes, and lines joing those points, called edges). Each node represents a guest, and a line between two guests represents a handshake. The *degree* of a node is the number of edges that connect to it, here it corresponds to the number of handshakes a guest took part in. Since every edge connects two nodes, if we sum the degree of every node, we count the number of edges twice, so

$$\sum_{v \text{ is a node}} \text{degree}(v) = 2\#(E)$$

   where $\#(E)$ means the number of edges.

   From our party's guests' claims, we see the sum of the degree of every guest is $5 \times 11 = 55$. This is not an even number, so cannot be twice the totaly number of handshakes - thus someone is lying.

5. This question is very similar, and uses the same rule we described above. Facebook is another graph, where users form the nodes and friendships form the edges. The sum of the degree of every node is then the sum of the number of all users' facebook friends. As we saw, this must be even. Let's divide the graph into two sets of nodes, V_o are the

users with an odd number of friends and V_e the users with an even number. If user v has an even number of friends we write $v \in V_e$ (read as v is in V_e). Then

$$\sum_{v \in V_e} \underbrace{\text{degree}(v)}_{\text{an even number}} + \sum_{v \in V_o} \underbrace{\text{degree}(v)}_{\text{an odd number}} = 2\#(E) = \text{an even number.}$$

Summing up even numbers, we get even numbers, but summing up odd numbers we get an odd number if there is an odd number of them (i.e. $\#(V_o)$ is odd) and we get an even number if there is an even number of them (i.e. $\#(V_o)$ is even). Since we must have an even number for the second term above, because the first term is even, then $\#(V_o)$ must be even.

(PS: This is called the *handshake lemma*)

6. Draw a line from B through P to some point S. Draw AS and SM. Draw AP which intersects SM at I. Draw a line from B through I to intersect AS at Q. Then QP is parallel to AB. To see this, relies on Ceva's Theorem:

   Given a triangle ABC, let the lines AO, BO, CO be drawn from the vertices through a common point O and meet opposite sides at D, E and F respectively. Then, using signed lengths of segments (so AB is positive if B is to the right of A) we have

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.$$

   Applying this to the above

$$\begin{aligned} \frac{AM}{MB} \times \frac{BP}{PS} \times \frac{SQ}{QA} &= 1 \\ \frac{BP}{PS} &= \frac{QA}{SQ} = \frac{AQ}{QS} \\ \frac{BP}{PS} + 1 &= \frac{AQ}{QS} + 1 \\ \frac{BP + PS}{PS} &= \frac{AQ + QS}{QS} \\ \frac{BS}{PS} &= \frac{AS}{QS}. \end{aligned}$$

   Then triangles ASB and QSP are similar, as they have a common angle and two sides in ratio. Thus QP is parallel to AB.

7. (This question was from *Parabola* volume 23, number 1. Here I will paraphrase their solution) Construct a new sequence by $d_n^2 = 2^{n+1} - 7c_{n-1}^2$. With a little exploration, we can compute some values for d_n, up to a choice in sign (since we have only specified its square). Happily enough, it turns out we can choose the signs so that $d_n = -d_{n-1} - 2d_{n-2}$ - the same recursive formula that the c_n follow.

   This connection runs depper, and indeed $d_n = 2c_n + c_n - 1$. Now it remains to prove that $2^{n+1} - 7c_{n-1}^2 = d_n^2 = (2c_n + c_{n-1})^2$ for $n \geq 2$. We will use induction.

For $n = 2$, $(2^3 - 7c_1^2) = 8 - 7 = 1$ and $(2c_2 + c_1)^2 = (-2 + 1)^2 = 1$. Now suppose it's true for $n = k$, then

$$\begin{aligned}(2c_{k+1} + c_k)^2 &= (2(-c_k - 2c_{k-1} + c_k)^2 = (-c_k - 4c_{k-1})^2 \\ &= c_k^2 + 8c_k c_{k-1} + 16c_{k-1}^2 \\ &= 14c_{k-1}^2 - yc_k^2 + 2(4c_k^2 + 4c_k c_{k-1} + c_{k-1}^2) \\ &= 14c_{k-1}^2 - 7c_k^2 + 2(2c_k + c_{k-1})^2 \\ &= 14c_{k-1}^2 - 7c_k^2 + 2(2^{k+1} - 7c_{k-1}^2) \\ &= 2^{k+2} - 7c_k^2.\end{aligned}$$

An alternative solution is also offered, and be sourced by going to `www.parabola.unsw.edu.au`, clicking Volume 28, Issue 2 and then opening up the "Solutions to Problems 861–871" section.

Senior Questions

Consider an equilateral triangular hole, and the piece that fits into it. The *symmetry group* of an equilateral triangle is made up of the operations you can do to the piece so that it still fits in its hole. For instance, you can rotate it by $60°$.

1. First we can do nothing, call this e. There are 2 rotations - by $60°$ and $120°$ in a clockwise direction, call them r_{60} and r_{120}. Finally there are 3 flips, about each of the 3 lines of symmetry, call them f_1, f_2 and f_3 where for f_i the vertex labelled i stays in the same spot.

2. Let our triangle's vertices be labelled, from the top in a clockwise direction (123). Then $r_{60}(123) = (312)$, and $f_1(123) = (132)$. Let's put these together $r_{60}(f_1(123)) = r_{60}(132) = (213)$ and $f_1(r_{60}(123)) = f_1(312) = (321)$. These are not the same, so they don't necessarily commute.

3. There's no coincidence I labelled e as I did. The "doing nothing" operation is precisely the identity. Doing a rotation, then nothing is the same as doing nothing then a rotation, which is the same as just doing the rotation.

4. Notice that the inverse of x can indeed be x - the most obvious case of this is when $x = e$, but you can also see it with the flips - doing a flip twice means you're back to where you started.

Operation	**Inverse**
e	e
r_{60}	r_{120}
r_{120}	r_{60}
f_1	f_1
f_2	f_2
f_3	f_3