MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 7, June 17, Brazil 2014[1]

1. Let's simply start by substituting x into the equation given, so

$$\begin{aligned}
x^3 - 3x\sqrt[3]{60} &= \left(\sqrt[3]{10} + \sqrt[3]{6}\right)^3 - 3\left(\sqrt[3]{10} + \sqrt[3]{6}\right)\sqrt[3]{60} \\
&= \left(10 + 3\sqrt[3]{10^2}\sqrt[3]{6} + 3\sqrt[3]{10}\sqrt[3]{6^2} + 6\right) - 3\left(\sqrt[3]{600} + \sqrt[3]{360}\right) \\
&= 10 + 3\sqrt[3]{600} + 3\sqrt[3]{360} + 6 - 3\sqrt[3]{600} - 3\sqrt[3]{360} \\
&= 16.
\end{aligned}$$

Now we can consider the polynomial $p(x) = x^3 - 3x\sqrt[3]{60} - 16$. Our value $x = \sqrt[3]{10} + \sqrt[3]{6}$ is a zero of p, and our task – to show that $x < 4$ – is to show that there are no zeros of p greater than or equal to 4. To do such, we can show that the graph, $y = p(x)$ is either positive and rising, or negative and falling for $x \geq 4$. So let's take a look at $p(4)$:

$$\begin{aligned}
p(4) &= 4^3 - 3(4)\sqrt[3]{60} - 16 \\
&= 64 - 16 - 12\sqrt[3]{60} \\
&= 48 - 12\sqrt[3]{60}.
\end{aligned}$$

Now $3^3 = 27$ and $4^3 = 64$ so $\sqrt[3]{60}$ is somewhere between 3 and 4. Thus $48 - 12\sqrt[3]{60} > 48 - 12 \times 4 = 0$, so $p(4) > 0$. To show now that p is rising for $x \geq 4$ we take it's derivative and solve for $p'(x) \geq 0$

$$\begin{aligned}
p'(x) = 3x^2 - 3\sqrt[3]{60} &\geq 0 \\
x^2 &\geq \sqrt[3]{60}
\end{aligned}$$

and again, since $4 > \sqrt[3]{60}$ then we certainly have $p'(x) \geq 0$ when $x^2 > 4$ or $x > 2$. So $p(4) > 0$ and for $x > 4$, $p'(x) \geq 0$ which means there cannot be any zeros of p above 4, so all zeros, $\sqrt[3]{10} + \sqrt[3]{6}$ included, are less than 4.

No doubt there is a way to show this without using calculus, if you have your own solution I'd love to see it.

[1] Some problems from UNSW's publication *Parabola*, the Mathematics Enrichment club is in no way affiliated with FIFA or the greatest sport in the world.

2. - In the case where the deleted vertex is part of a triangle, two vertices of the triangle merge to become one, so $v \to v-1$, two edges disappear, the one connecting the two merged vertices disappears while the other two merge to become one, so $e \to e-2$ and finally, the face formed by the triangle disappears so $f \to f-1$. If the deleted vertex is not part of a triangle, its either part of a lone line, or a higher order polygon. If a lone line, then we simply see one vertex disappear and the edge that connected the two along with it, so $f \to f$, $v \to v-1$ and $e \to e-1$. If part of a higher order polygon, again the number of faces stays the same and we just lose one edge and one vertex, so again $f \to f$, $v \to v-1$ and $e \to e-1$.
   - As we can see from the two cases above, $v-e+f$ is constant when we delete a vertex by merging it with a neigbour (in the first case $v-e+f \to v-1-(e-2)+f-1 = v-e+f$ and in the second $v-e+f \to v-1-(e-1)+f = v-e+f$). So we can delete all but one vertex in this manner, all while $v-e+f$ is constant. A graph on a sphere with only one vertex has $v = 1$, $e = 0$ and $f = 1$ so $v-e+f = 2$.
   - Suppose we have b pentagons and w hexagons. The number of faces is $f = b+w$. The number of edges are $5b$ from the pentagons and $6w$ from the hexagons, but we've counted each edge twice (as each edge belongs to the polygon on either side of it), so $e = (6w+5b)/2$. Finally, we can count the vertices in a similar fashion – $5b$ from the pentagons and $6w$ from the hexagons, but each vertex belongs to the 3 adjacent polygons, so $v = (6w+5b)/3$.

     From the previous part, we know that $v-e+f = 2$, so

     $$\begin{aligned} 2 &= (6w+5b)/3 - (6w+5b)/2 + b + w \\ &= 2w + \frac{5}{3}b - 3w - \frac{5}{2}b + b + w \\ 12 &= 10b - 15b + 6b = b. \end{aligned}$$

     So there are 12 pentagons. Now each pentagon meets only hexagons, so say 5 hexagons belong to each pentagon. However, each hexagon meets alternately pentagons and hexagons, so each hexagon belongs to 3 pentagons. Thus $w = \frac{5}{3}b = 20$. So there are 12 pentagons and 20 hexagons on a standard soccer ball.

3. The key to this question is finding under what conditions Australia makes the top two, and computing their probabilities. There are 81 possible outcomes of the next four games, so doing this problem by "brute force" (computing the probability for each possible outcome and adding up those in which Australia advances) would be quite the arduous task. Instead, we should look for some simplifications. For instance, if Australia wins their next two games they're guaranteed a spot in the top two, regardless of what happens to the other teams. This takes care of 9 out of the total 81 possible outcomes for us. Similarly if they lose the next two games there's no chance of advancing, removing another 9 possibilities.

   For Australia to get through on 3 points, they must do it by beating the Netherlands, have Chile win all 3 games, then win the coin toss against the Netherlands or by beating Spain, have Chile lose their next two games, then win the coin toss against Spain and Chile.

Australia cannot advance with 2, 1 or 0 points.

The most intensive situation for us is if Austrlia gets 4 points, in which the short answer is maybe.

We can start by writing down an upper bound for the probability:

$$\begin{aligned}\mathbb{P}(\text{Australia advances}) &< \mathbb{P}(\text{Aust gets 6 points}) + \mathbb{P}(\text{Aust gets 4 points}) + \mathbb{P}(\text{Aust gets 3 points}) \\ &= \left(\frac{1}{13}\frac{1}{15}\right) + \left(\frac{1}{13}\frac{1}{10} + \frac{8}{65}\frac{1}{15}\right) + \left(\frac{1}{13}\frac{5}{6} + \frac{4}{5}\frac{1}{15}\right) \\ \mathbb{P}(\text{Australia advances}) &< \frac{9}{65}.\end{aligned}$$

We could then refine this by including the appropriate conditional probabilities, for instance the probability that Australia advances by "beating the Netherlands, have Chile win all 3 games, then win the coin toss against the Netherlands" is

$$\mathbb{P} = \mathbb{P}(A \text{ beats } N)\mathbb{P}(S \text{ beats } A)\mathbb{P}(C \text{ beats } S)\mathbb{P}(C \text{ beats } N) \times \frac{1}{2}$$

which makes up one part of $\mathbb{P}(\text{Australia advances with 3 points}) < \mathbb{P}(\text{Australia gets 3 points})$.

4. This question was taken from UNSW's *Parabola* but the solution is missing a diagram, the website `http://mathafou.free.fr/pbg_en/sol110b.html` does an excellent job of showing the same construction with an accompanying diagram.

5. The two triangles ΔKXA and ΔDXC are similar as the three corresponding angles are equal. Suppose the $AK = \ell$ then since K is the midpoint of AB, $DC = 2\ell$. Let the height of rectangle, BC, be h, and the area of $\Delta KXA = a$. Since ΔKXA and ΔDXC are similar with side ratios 1 : 2 then they have areas in ratio 1 : 4. If the perpendicular height of ΔKXA is y, then the sum of their areas can also be written as $\frac{1}{2}\ell y + \frac{1}{2}(2\ell)(h-y)$ so

$$\begin{aligned} a + 4a &= \frac{1}{2}\ell y + \frac{1}{2}(2\ell)(h-y) \\ &= \frac{1}{2}\ell y + \ell h - \ell y \\ &= \ell h - \frac{1}{2}\ell y = \ell h - a \\ 6a &= \ell h \\ 12a &= (2\ell h) \end{aligned}$$

where $2\ell h$ is precisely the area of the rectangle. So ΔKXA takes up 1/12th the area of $ABCD$.

Senior Questions

1. This is known as the *intermediate value theorem* and is essentially the result of the real numbers 'having no gaps', and that continuous functions can't 'jump'. What we

mean by continuous functions not being able to 'jump' is that if x_1 and x_2 are close together, then so are $f(x_1)$ and $f(x_2)$. In symbols, we say if $|x_1 - x_2| < \delta$ then there's a ϵ so that $|f(x_1) - f(x_2)| < \epsilon$ and vice versa (the distance between x_1 and x_2 being smaller than a certain number implies that the distance between $f(x_1)$ and $f(x_2)$ is less than another number and we can write a rule that tells you the other if you know one).

Let's choose a z between c and d and call S all the numbers between a and b which satisfy $f(x) \leq z$. Now consider the numbers that are greater than or equal to every number in S, call the smallest of these y. I'm going to show you that $f(y) = z$.

Suppose $|f(x) - f(y)| < \epsilon$ then there's a δ with $|x - y| < \delta$. This means that

$$\epsilon - f(x) < f(y) < \epsilon + f(x)$$

for all x between $y - \delta$ and $y + \delta$. There are numbers between $y - \delta$ and y that are in S (because if there weren't, there'd be a smaller value to choose for y which means we didn't choose the smallest). So for those x

$$f(y) < \epsilon + f(x) \leq \epsilon + z$$

because $f(x) \leq z$.

All the x between y and $y + \delta$ are not in S, so here we have $f(x) > z$, i.e.

$$f(y) > \epsilon - f(x) \geq \epsilon - z$$

so bringing these together

$$\epsilon - z < f(y) < \epsilon + z.$$

Now all this works for any positive number ϵ, because that's what makes the function, f, continuous. So if we choose ϵ exceedingly small, we see that this only works if $f(y) = z$.

2. Rather than looking at the function f, we'll consider the function $g(x) = f(x) - x$. If g has a zero, then f has a fixed point. Note that g is continuous because it is the sum of two continuous functions.

   Now $g(a) = f(a) - a \geq 0$ since $a \leq f(x)$ for all x between a and b. Similarly $g(b) = f(b) - b \leq 0$ because $f(x) \leq b$ for all x between a and b. So either g is zero at either $x = a$ or $x = b$ or g goes from positive to negative, and since it is continuous the intermediate value theorem says it has to have crossed 0 at some point.

   That is, we know that for some c between a and b, $g(c) = 0$ which means $f(c) - c = 0 \implies f(c) = c$.

3. If g is a constant function, then $g'(x) = 0$ for all x between a and b. So suppose g is not constantand $g'(a) > 0$ (or g is increasing at a). Now if $g'(x) > 0$ for all x between a and b then it is not possible for $g(a) = g(b)$, so there must be some other number $a < b' < b$ for which $g'(b') \leq 0$. If $g'(b') = 0$ we've found our c, and if not, then we can use the intermediate value theorem (assuming $g'(x)$ is a continuous function) to see that there must be a c between a and b' with $g'(c) = 0$.

   Can you extend this to the case where g' is not necessarily continuous?

4. This was supposed to read "Let g be as above except $g(a) \neq g(b)$" otherwise it's essentially the same as the previous question.

   Let $g(a) = s$ and $g(b) = t$, and consider the function $h(x) = g(x) - \left(\frac{t-s}{b-a}(x-a) + s\right)$. Note that $h(a) = g(a) - s = 0$ and $h(b) = g(b) - t = 0$. Also, h is the sum of two continuous and differentiable functions so the theorem of the previous question applies, which means there's a c such that $h'(c) = 0$, or rather

$$g'(c) - \left(\frac{t-s}{b-a}\right) = 0 \implies g'(c) = \frac{g(b) - g(a)}{b-a}.$$