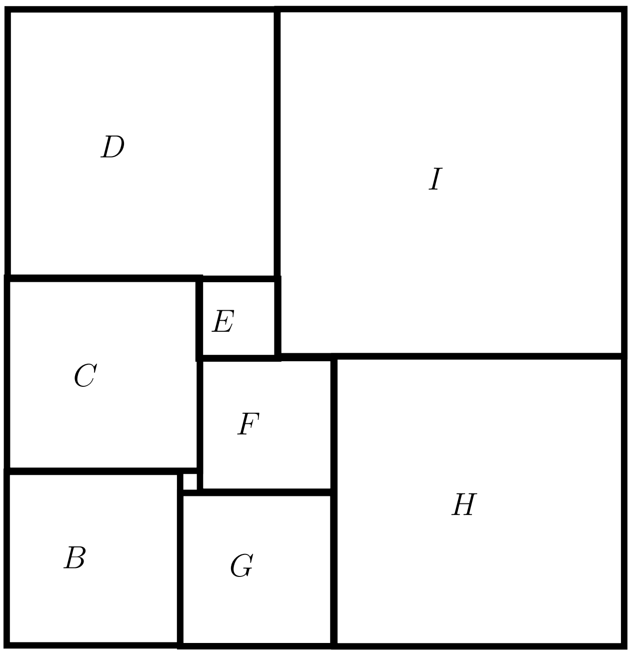
MATHEMATICS ENRICHMENT CLUB.
Hint Sheet 11, August 5, 2014[1]

1. If cog 1 rotates clockwise, cog two must rotate counter clockwise, and so cog three must rotate clockwise and so on. All the odd cogs must rotate the same way.



2. We can work out the side lengths for each square. Let the lowercase letter for a square be its side length, then say $b = 1 + g$, $g = 1 + f$, $c = 1 + b$, $d = c + e$ and so on. The total area is $(d + c + b) \times (b + g + h) = (d + i) \times (i + h)$. With a bit of work we can determine that the total area is 1056.

3. The probability of a coin flip coming up heads is $1/2$.

   To get a big loop of string it doesn't matter how the tops are tied. There are 6 ends at the bottom, so the total number of ways of tying them into pairs is $\binom{6}{2}$, how many of these result in one big loop? Take one end, it can be tied to 5 other ends, one of which results in two pieces of string becoming its own loop (the piece of string it's tied to at the top). So tie it to one of the other 4. After this, there are 4 remaining ends

[1] Some problems from UNSW's publication *Parabola*, others from `www.brilliant.org`

– choose one, then it can be tied to 3 other ends, but one of those either results in a loop with 2 pieces of string, or a loop with 4 pieces of string, so tie it to one of the other two. Thus there are 4×2 ways of making one big loop once the tops are tied. Meaning the probability of you winning this game is $1 - 8/15 = 7/15 < 1/2$.

Thus you're more likely to win the coin game.

4. The ant should walk in straight lines to walk the shortest distance. The shortest path from A to the line BX is that which is perpendicular to BX and through A, let's call the length of this path s. Then the shortest distance from A to C is $2s$. It remains to find s, which can be done using the triangle ΔAXB which is isoceles, and s is the altitude from A. Immediately we can spot some restrictions, for instance if ΔAXB is right angled or larger at X, then the shortest path is to go right over the top of the pyramid. If the pyramid is taller, one must calculate s – the algebra is long and arduous, but possible, as we know all three sides of ΔAXB and just need to calculate the altitude sitting on XB.

5. We must $5^n \mod 13 + n^5 \mod 13 = 0 \mod 13$, so write out $5^n \mod 13$ for various n and $n^5 \mod 13$ for various n (both sequences will eventually repeat themselves). Then find the n for which both sequences add up to a mulitple of 13. The first one is $n = 12$, but can you figure out how to generate all of them?

6. Think about it this way, on Wednesday, if you flip a HTT you need at least 3 more flips to get HTH, but on Thursday if you flip HTH you only need two more flips to get HTT as you've already got the first H.