**MATHEMATICS ENRICHMENT CLUB.**
**Solution Sheet 1, May 5, 2015**

1. Let $x = 0.284284284\ldots$, then

$$\begin{aligned}1000x &= 284.284284284\ldots\\ &= 284 + x,\end{aligned}$$

thus $x = 284/999$.

2. We can write the finite sum as

$$1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{10100} = 1 + \sum_{n=2}^{101} \frac{1}{n \times (n-1)}$$

Using the given formula,

$$\begin{aligned}1 + \sum_{n=2}^{101} \frac{1}{n \times (n-1)} &= 1 + \sum_{n=2}^{101} \frac{n-1}{n} - \frac{n-2}{n-1}\\ &= 1 + \sum_{n=2}^{101} \frac{n-1}{n} - \sum_{n=1}^{100} \frac{n-1}{n}\\ &= 1 + \frac{100}{101}.\end{aligned}$$

3. Let $S$ be the number of members that plays Soccer.

(a) If we add the number of members that plays either Basketball, Cricket or Soccer, we would end up with a number that is greater than the total number of members in the sports club, because we have double counted the number of members that plays two sports *only*, and triple counted the number of members that plays *all* three.

So to balance this out we need to subtract the double/triple counts: We know that 10 plays *all* three sports, so these member we triple counted. There is 60 members that plays two or more sports, and 10 that plays all three, therefore there is $60 - 10 = 50$ members that plays two sports *only*.

The balanced equation is then

$$163 = S + 100 + 73 - 50 - 2(10),$$

which gives $S = 60$.

(b) The number of members that plays both Basketball and Cricket but not Soccer is $25 - 10 = 15$, therefore $60 - 15 = 45$ members plays Soccer and Basketball or Soccer and Cricket or all three sports. Since $S = 60$, $60 - 45 = 15$ of these members plays Soccer only.

4. (a) Here $|QC|$ means the length of $QC$. By construction, the length of $AP$ is $b$; that is$|AP| = b$. Since the point $Q$ is the intersection of the tangent $PQ$ and $CQ$ of the same circle arc $PC$, $|PQ| = |CQ|$ (you may want to prove this as an exercise). So the problem is reduced to finding $|PQ|$. Note that $BPQ$ is an isosceles triangle, so $|CQ| = |PQ| = |PB| = a - b$.

   (b) Suppose we can find whole numbers $a$ and $b$ such that the ratio $a/b$ is in its simplest form and $a/b = \sqrt{2}$. We'll use the result of part $a$) to produce a contradiction on the condition that the ratio $a/b$ is in its simplest forms.

   As noted in part $a$), the triangle $BPQ$ is an isosceles triangle. Moreover, $BPQ$ is similar to the triangle $ABC$; so that the ratio

$$\frac{|AB|}{|AC|} = \frac{|BQ|}{|BP|} = \frac{|BC| - |QC|}{|BA| - |PA|},$$

   holds. Since $ABC$ is a right angled isosceles triangle, we have$|AB|/|AC| = \sqrt{2} = a/b$. So the above ratio equation can be expressed in terms of $a$ and $b$ as

$$\frac{a}{b} = \frac{b - (a - b)}{a - b},$$

   so we have just found $\sqrt{2} = (2b - a)/(a - b)$. Now we can argue that if $a$ and $b$ are whole numbers, then $2b - a$ is a whole number smaller than $a$, and $a - b$ is a whole number smaller than $b$, this contradicts $a/b$ being in the simplest form.

5. (a) Let $a_1, a_2, \ldots a_k$ be the digits of a $k$ digit long whole number $x$. Then we can expression $x = a_k a_{k-1} \ldots a_2 a_1 = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_3 + a_2 a_1$. Since $10^i$ is divisible by 4 for $i = 2, 3, \ldots k$, for $x$ to be divisible by 4, then so is $a_2 a_1$; the number formed by the last two digits of $x$.

   (b) Similar to $a$), first express the $k$ digit long number as $x = a_k a_{k-1} \ldots a_2 a_1 = 10^k a_k + 10^{k-1} a_{k-1} + \ldots + 10^2 a_3 + 10 a_2 + a_1$. Let $y$ be the number formed by the sum of all of the digits of $x$; that is $y = a_1 + a_2 + \ldots + a_{k-1} + a_k$. Consider the difference $x - y = (10^k - 1) a_k + (10^{k-1}) a_{k-1} + \ldots + 99 a_3 + 9 a_2$, then $x - y$ is divisible by 9, so if $x$ is divisible by 9, then so must $y$.

6. Let $x = n(n+1)(n+2)$, then $x$ is the product of 3 consecrative numbers. Hence, $x$ is divisible by 2 and 3, which means $x^2$ is a divisible by 4 and 9. We can now use the results of Q5 to find the missing digit $*$:

   Since $x^2$ is divisible by 9, the result of 5.$a$) says that the sum of the digits of $x^2$; that is $4 + 8 + 1 + 2 + 7 + 3 + 5 + 6 + 3 + * + 6 = 45 + *$ must be divisible by 9, so the digit $*$ must be either 0 or 9.

   Also, $x^2$ is divisible by 4, so by the result of 5.$b$) the number $*6$ formed by the last two digits of $x^2$ must be divisible by 4, this can only happen if $*6 = 40$, therefore $* = 0$.

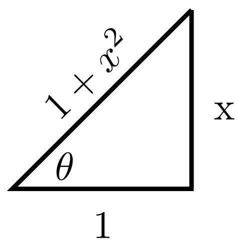
**Senior Questions**

1. Let $\theta = \tan^{-1}(x)$, then $x = \tan(\theta)$, and

$$\frac{dx}{d\theta} = \frac{1}{\cos^2\theta}.$$

Now by using the picture below $\cos^2\theta = 1/(1+x^2)$, therefore $dx/d\theta = 1 + x^2$, thus

$$\frac{d\theta}{dx} = \frac{d}{dx}\tan^{-1}(x) = \frac{1}{1+x^2}$$




2. By polynomial long division (see for example `http://en.wikipedia.org/wiki/Polynomial_long_division`), we can express

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \dots.$$

Since the LHS of the above equation is $\frac{d}{dx}\tan^{-1}(x)$ by Q1, integrating both sides gives

$$\tan^{-1}(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

3. This solution provide by Adam Solomon:

$$\begin{aligned} n(n+1)(n+2)(n+3)+1 &= (n^2+3n)(n^2+3n+2)+1 \\ &= (n^2+3n)(n^2+3n+1)+n^2+3n+1 \\ &= (n^2+3n+1)^2. \end{aligned}$$

To calculate $\sqrt{(31)(30)(29)(28)+1}$, set $n = 28$ in $n^2 + 3n + 1$.