

## TRIGONOMETRIC IDENTITIES

In order to work effectively with trigonometric functions, you need to know **all** of the following basic identities.

- Periodicity:

$$\cos \theta = \cos(\theta + 2\pi) , \quad \sin \theta = \sin(\theta + 2\pi) .$$

- Symmetry (even and odd functions):

$$\cos(-\theta) = \cos \theta , \quad \sin(-\theta) = -\sin \theta .$$

- Pythagoras' theorem (trigonometric version):

$$\cos^2 \theta + \sin^2 \theta = 1 .$$

- Other functions:

$$\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \text{and} \quad \sec \theta = \frac{1}{\cos \theta} ,$$

provided  $\theta \neq (n + \frac{1}{2})\pi$  with  $n$  an integer;

$$\cot \theta = \frac{\cos \theta}{\sin \theta} \quad \text{and} \quad \csc \theta = \frac{1}{\sin \theta} ,$$

provided  $\theta \neq n\pi$  with  $n$  an integer.

- Addition formulae:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta . \end{aligned}$$

- Double angle formulae:

$$\begin{aligned} \cos 2\alpha &= \cos^2 \alpha - \sin^2 \alpha \\ &= 2 \cos^2 \alpha - 1 \\ &= 1 - 2 \sin^2 \alpha \\ \sin 2\alpha &= 2 \sin \alpha \cos \alpha . \end{aligned}$$

- Exact values:

$$\begin{aligned} \cos 0 &= 1 , \quad \cos\left(\frac{\pi}{2}\right) = 0 , \quad \cos \pi = -1 , \\ \sin 0 &= 0 , \quad \sin\left(\frac{\pi}{2}\right) = 1 , \quad \sin \pi = 0 . \end{aligned}$$

You should also know the values of  $\cos\left(\frac{\pi}{6}\right)$ ,  $\sin\left(\frac{\pi}{4}\right)$  and so on. See a separate revision worksheet for these.

You also need to be able to use the above formulae to derive others. For example, take the addition formula for  $\cos$ , replace  $\beta$  by  $-\beta$ , and use the fact that  $\cos$  is even and  $\sin$  is odd:

$$\begin{aligned} \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta . \end{aligned}$$

Now add these two equations,

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta , \quad (*)$$

to obtain a formula for a product of cosines,

$$\cos \alpha \cos \beta = \frac{\cos(\alpha + \beta) + \cos(\alpha - \beta)}{2} .$$

Remember, you **do not** need to memorise this, you need to know how to work it out.

## EXERCISES.

Please try to complete the following exercises. Remember that you **cannot** expect to understand mathematics without doing lots of practice! Please do not look at the answers before trying the questions. If you get a question wrong you should go through your working carefully, find the mistake and fix it. If there is a mistake which you cannot find, or a question which you cannot even start, please consult your tutor or the Mathematics Drop-in Centre.

1. Use the formulae on pages 1–2 (and other basic algebraic identities) to prove that

(a)  $(\cos \theta + \sin \theta)^2 = 1 + \sin 2\theta$ ;

(b)  $\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta$ ;

(c)  $\sin^2(\alpha + \beta) - \sin^2(\alpha - \beta) = \sin(2\alpha) \sin(2\beta)$ .

2. Divide both sides of  $\cos^2 \theta + \sin^2 \theta = 1$  by  $\cos^2 \theta$  and hence find a relation between  $\tan \theta$  and  $\sec \theta$ .

3. Prove that for any  $\theta$  we have

$$\sin(\pi - \theta) = \sin \theta, \quad \cos(\pi - \theta) = -\cos \theta;$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta, \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta.$$

4. Find a formula for  $\tan(\alpha + \beta)$  by means of the following procedure: write down the definition of  $\tan(\alpha + \beta)$ ; use addition formulae to expand the numerator and denominator; divide numerator and denominator by  $\cos \alpha \cos \beta$ ; simplify.

5. “*Products-to-sums*” formulae. Take formulae for  $\cos(\alpha + \beta)$  and  $\cos(\alpha - \beta)$  as on page 2; subtract them instead of adding; hence find a formula for  $\sin \alpha \sin \beta$ . Starting with  $\sin(\alpha + \beta)$ , do something similar to find a formula for  $\sin \alpha \cos \beta$ ,

6. “*Sums-to-products*” formulae. If  $x = \alpha + \beta$  and  $y = \alpha - \beta$ , find  $\alpha, \beta$  in terms of  $x, y$ . By substituting into (\*), find a formula for  $\cos x + \cos y$ . Use similar ideas to find formulae for  $\cos x - \cos y$  and  $\sin x + \sin y$ .

## ANSWERS.

1. (b) *Hint.* Start with  $\cos 3\theta = \cos(2\theta + \theta)$ .

2.  $1 + \tan^2 \theta = \sec^2 \theta$ .

4. We have

$$\begin{aligned} \tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta - \sin \alpha \sin \beta} \\ &= \frac{\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta}}{1 - \frac{\sin \alpha \sin \beta}{\cos \alpha \cos \beta}} \\ &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \end{aligned}$$

5. The other two “products-to-sums” formulae are

$$\sin \alpha \sin \beta = \frac{\cos(\alpha - \beta) - \cos(\alpha + \beta)}{2},$$

$$\sin \alpha \cos \beta = \frac{\sin(\alpha + \beta) + \sin(\alpha - \beta)}{2}.$$

6. We obtain  $\alpha = \frac{x + y}{2}$ ,  $\beta = \frac{x - y}{2}$  and so

$$\cos x + \cos y = 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right),$$

$$\cos x - \cos y = -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right),$$

$$\sin x + \sin y = 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right).$$